PROPERTIES OF COUNTABLE SEPARATION AND IMPLICIT FUNCTION THEOREM (*)

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We consider some properties of countable separation for families $\mathcal{A}$ and $\mathcal{D}$ of subsets of a set $X$ by means of elements of a fixed family $\mathcal{M} \subseteq P(X)$. We give necessary and sufficient conditions, in terms of measurability of some sets constructed by means of multifunctions, in order that the families $\mathcal{A}$ and $\mathcal{D}$ satisfy such a property. As an application we derive an implicit function theorem for functions of Carathéodory-type.

Di Bari in a previous paper [4] introduced the notions of countably $\mathcal{A}$-$\mathcal{D}$-separated topological space and weakly countably $\mathcal{A}$-$\mathcal{D}$-separated topological space; she obtained a characterization of topological spaces of this type in terms of measurability of some sets constructed by means of multifunctions. In [4] the elements of the families $\mathcal{A}$ and $\mathcal{D}$ of subsets of a topological space $(X, \tau)$ are separated by means of elements of a countable family of open sets.

In this paper the topology in $X$ is replaced by a family $\mathcal{M}$ of subsets of $X$ and the elements of $\mathcal{A}$ and $\mathcal{D}$ are separated by means of elements of a countable subfamily of $\mathcal{M}$. We introduce two concepts of countable $s\mathcal{A}$-$\mathcal{D}$-separation and we give conditions in order that $(X, \mathcal{M})$ is of this

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type. If \( \mathcal{M} \) is a topology in \( X \), then we obtain results contained in \([1], [4]\) and if \( \mathcal{M} \) is a \( \sigma \)-algebra results contained in \([3], [5], [8]\). We prove, also, an implicit theorem for functions of Carathéodory-type that is an extension of Theorem 4.5 of \([7]\). In order to establish this theorem we make use of the notion of \( \mathcal{P} \)-system.

1. Some preliminaries.

Let \( T \) and \( X \) be sets, a multifunction \( F: T \to X \) is a function from \( T \) to \( \mathcal{P}(X) \), the power set of \( X \). We denote with \( \text{Gr}(F) \) the graph of \( F \), that is, the set \( \{(t, x) \in T \times X : x \in F(t)\} \) and, if \( U \subset X \), with \( F^+(U) \) the set \( \{t \in T : F(t) \subset U\} \).

If \( B \subset \mathcal{P}(X) \), we write \( \sigma(B) \) for the \( \sigma \)-algebra on \( X \) generated by \( B \). If \( A, B \subset \mathcal{P}(X) \) and \( U \subset X \) we denote with \( \langle U \rangle_A \) the set \( \{A \in A : A \subset U\} \), with \( \langle B \rangle_A \) the set \( \{\langle U \rangle_A : U \in B\} \) and with \( \sigma(A, \langle B \rangle) \) the \( \sigma \)-algebra on \( A \) generated by the family \( \langle B \rangle_A \).

We call \((X, \mathcal{M})\) a measurable space iff \( X \) is a non empty set and \( \mathcal{M} \) a \( \sigma \)-algebra on \( X \). The measurable space \((X, \mathcal{M})\) is countably separated if there exists a countable family \( \mathcal{H} \subset \mathcal{M} \) which separates the points of \( X \), that is, for any two distinct points \( x \) and \( y \) in \( X \) there is \( H \in \mathcal{H} \) with \( x \in H \) and \( y \notin H \), or, \( x \notin H \) and \( y \in H \).

If \( Y \) is a set and \( G: Y \to X \) a multifunction we write \( \text{Gr}(F, G) \) for the set \( \{(t, y) \in T \times Y : G(y) \subset F(t)\} \). Besides, if \((T, \mathcal{M}_1)\) and \((Y, \mathcal{M}_2)\) are measurable spaces with \( \mathcal{M}_1 \times \mathcal{M}_2 \) we denote the product \( \sigma \)-algebra on \( T \times Y \).

2. Some properties of separation.

Throughout this paper, unless otherwise indicated, \( X \) is a set and \( \mathcal{M}, \mathcal{A}, \mathcal{D} \) non empty families of subsets of \( X \). The purpose of this paper is that of introducing for \((X, \mathcal{M})\) some properties of separation, which can be