PROPERTIES OF COUNTABLE SEPARATION
AND IMPLICIT FUNCTION THEOREM (*)

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We consider some properties of countable separation for families \( A \) and \( D \) of subsets of a set \( X \) by means of elements of a fixed family \( M \subset \mathcal{P}(X) \). We give necessary and sufficient conditions, in terms of measurability of some sets constructed by means of multifunctions, in order that the families \( A \) and \( D \) satisfy such a property. As an application we derive an implicit function theorem for functions of Carathéodory-type.

Di Bari in a previous paper [4] introduced the notions of countably \( A-D \)-separated topological space and weakly countably \( A-D \)-separated topological space; she obtained a characterization of topological spaces of this type in terms of measurability of some sets constructed by means of multifunctions. In [4] the elements of the families \( A \) and \( D \) of subsets of a topological space \((X, \tau)\) are separated by means of elements of a countable family of open sets.

In this paper the topology in \( X \) is replaced by a family \( M \) of subsets of \( X \) and the elements of \( A \) and \( D \) are separated by means of elements of a countable subfamily of \( M \). We introduce two concepts of countable \( sA-D \)-separation and we give conditions in order that \((X, M)\) is of this

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type. If $M$ is a topology in $X$, then we obtain results contained in [1], [4] and if $M$ is a $\sigma$-algebra results contained in [3], [5], [8].

We prove, also, an implicit theorem for functions of Carathéodory-type that is an extension of Theorem 4.5 of [7]. In order to establish this theorem we make use of the notion of $T$-system.

1. Some preliminaries.

Let $T$ and $X$ be sets, a multifunction $F: T \to X$ is a function from $T$ to $\mathcal{P}(X)$, the power set of $X$. We denote with $Gr(F)$ the graph of $F$, that is, the set $\{(t, x) \in T \times X : x \in F(t)\}$ and, if $U \subset X$, with $F^+(U)$ the set $\{t \in T : F(t) \subset U\}$.

If $B \subset \mathcal{P}(X)$, we write $\sigma(B)$ for the $\sigma$-algebra on $X$ generated by $B$. If $A, B \subset \mathcal{P}(X)$ and $U \subset X$ we denote with $\langle U \rangle_A$ the set $\{A \in A : A \subset U\}$, with $\langle B \rangle_A$ the set $\{\langle U \rangle_A : U \in B\}$ and with $\sigma(A, \langle B \rangle)$ the $\sigma$-algebra on $A$ generated by the family $\langle B \rangle_A$.

We call $(X, M)$ a measurable space iff $X$ is a non empty set and $M$ a $\sigma$-algebra on $X$. The measurable space $(X, M)$ is countably separated if there exists a countable family $\mathcal{H} \subset M$ which separates the points of $X$, that is, for any two distinct points $x$ and $y$ in $X$ there is $H \in \mathcal{H}$ with $x \in H$ and $y \notin H$, or, $x \notin H$ and $y \in H$.

If $Y$ is a set and $G: Y \to X$ a multifunction we write $Grc(F, G)$ for the set $\{(t, y) \in T \times Y : G(y) \subset F(t)\}$. Besides, if $(T, M_1)$ and $(Y, M_2)$ are measurable spaces with $M_1 \times M_2$ we denote the product $\sigma$-algebra on $T \times Y$.

2. Some properties of separation.

Throughout this paper, unless otherwise indicated, $X$ is a set and $M$, $A$, $D$ non empty families of subsets of $X$. The purpose of this paper is that of introducing for $(X, M)$ some properties of separation, which can be