JENSEN'S INEQUALITY FOR OPERATOR MONOTONE FUNCTIONS

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In this paper, we shall show that some classical inequalities for monotone functions also hold for operator monotone functions on an arbitrary Hilbert space \( H \).

Such results can be found in the classical book [1, p. 83] or in a new book [2].

Results.

THEOREM 1. A necessary and sufficient condition such that

\[
\left( \sum_{i=1}^{n} p_i \right) f \left( \sum_{i=1}^{n} A_i \right) \geq \sum_{i=1}^{n} p_i f(A_i)
\]

for all strictly positive operators \( A_i \) in \( H \) and strictly positive numbers \( p_i \) is that \( f(x) \) should be operator monotone for \( x > 0 \).

Strict inequality holds if \( f(x) \) is strictly operator monotone and there is more than one \( A \).

Proof. (i) If \( f \) is operator monotone, then

\[
f \left( \sum_{i=1}^{n} A_i \right) \geq f(A_i) \quad (i = 1, \ldots, n).
\]
Multiplying by \( p_i \), and adding all such inequalities gives (1).

(ii) If in (1), we take \( n = 2 \), \( A_1 = A \), \( A_2 = B \), \( p_1 = 1 \), \( p_2 = p \), we obtain

\[
(1 + p)f(A + B) \geq f(A) + pf(B)
\]

Letting \( p \to 0 \), we see that \( f(A + B) \geq f(A) \), so that \( f \) is monotone.

**Remark.** In fact, Theorem 1 holds if all \( p_i \) are positive operators, permutable with \( A \), and \( \sum_{i=1}^{n} A_i \).

**Theorem 2.** If for \( x > 0 \), the function \( x^{-1}f(x) \) is operator monotone for permutable operators, then for all strictly positive operators \( A_i \), \( i = 1, \ldots, n \) on \( H \) permutable with \( \sum_{i=1}^{n} A_i \), we have

\[
(2) \quad f \left( \sum_{i=1}^{n} A_i \right) \geq \sum_{i=1}^{n} f(A_i).
\]

There is strict inequality if \( x^{-1}f(x) \) is strictly operator monotone for permutable operators and there is more than one \( A_i \);

**Proof.** Since \( x^{-1}f(x) \) is operator monotone,

\[
\left( \sum_{i=1}^{n} A_i \right)^{-1} f \left( \sum_{k=1}^{n} A_k \right) \geq A_i^{-1} f(A_i) \quad (i = 1, \ldots, n).
\]

Multiplying \( A_i \), we get

\[
A_i \left( \sum_{k=1}^{n} A_k \right)^{-1} f \left( \sum_{k=1}^{n} A_k \right) \geq f(A_i).
\]

Summing over \( i = 1, \ldots, n \), gives (2).

**Remark.** Theorems 1 and 2 are operator versions of Theorems 102 and 103 from [1].

A generalization of theorem 2 can be given as follows: