HEREDITARY ORDER CONVEXITY IN $\mathcal{L}(X,Y)$

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Let $X$ be a topological vector space, $Y$ an ordered topological vector space and $\mathcal{L}(X,Y)$ the space of all linear and continuous mappings from $X$ into $Y$. The hereditary order-convex cover $[K]^h$ of a subset $K$ of $\mathcal{L}(X,Y)$ is defined by $[K]^h = \{ A \in \mathcal{L}(X,Y) : Ax \in [Kx] \text{ for all } x \in X \}$, where $[Kx]$ is the order-convex of $Kx$. In this paper we study the hereditary order-convex cover of a subset of $\mathcal{L}(X,Y)$. We show how this cover can be constructed in specific cases and investigate its structural and topological properties. Our results extend to the space $\mathcal{L}(X,Y)$ some of the known properties of the convex hull of subsets of $X^*$.

1. Introduction.

During the recent years, much attention has been attracted by the space $\mathcal{L}(X,Y)$ of all linear and continuous mappings from a topological vector space $X$ into an ordered topological vector space $Y$. This space is important to the study of subdifferentials of convex operators from $X$ into $Y$. However, the lack of suitable separability results in $\mathcal{L}(X,Y)$, analogous to those holding in $X^* = \mathcal{L}(X,\mathbb{R})$, but making use of the ordering of $Y$, soon became apparent (cf. [4], [5]). It seems furthermore that the notion of convexity is not appropriate to this aim. In [1] a notion of hereditary order convexity was introduced, which expresses the desired separability between sets and points and
which is closely related to convexity when \( Y = \mathbb{R} \). This notion was the basic tool for the derivation of some results concerning the study of monotone operators.

This paper is devoted to the study of the hereditary order convex cover of a subset of \( \mathcal{L}(X, Y) \). We show how this cover can be constructed in specific cases, and we investigate its structural and topological properties. In particular, our results extend to the space \( \mathcal{L}(X, Y) \) some of the known properties of the convex hull of subsets of \( X^* \).

2. Preliminaries.

Throughout this paper we shall denote by \( X \) a real locally convex Hausdorff space and by \( Y \) a real locally convex Hausdorff space which is also an ordered linear space with closed positive cone \( Y_+ \). Let \( \mathcal{L} = \mathcal{L}(X, Y) \) be the space of all linear and continuous mappings from \( X \) into \( Y \). We denote by \( \mathcal{L}_s(X, Y) \) the space \( \mathcal{L}(X, Y) \) endowed with the topology of simple convergence [3].

Let \( X^* \) be the dual space of \( X \) and \( \langle x^*, x \rangle \) the duality pairing between \( x^* \in X^* \) and \( x \in X \). In what follows, we shall always consider \( X^* \) endowed with the \( w^* \)-topology, which is nothing else but the topology of simple convergence, if we view \( X^* \) as \( \mathcal{L}(X, \mathbb{R}) \). If \( x^* \in X^* \) and \( y \in Y \) then \( x^* \otimes y \) represents the element \( A \in \mathcal{L}(X, Y) \) defined by

\[
Ax = \langle x^*, x \rangle y, \ x \in X.
\]

If \( K \subset \mathcal{L} \) and \( x \in X \), we denote by \( Kx \) the set \( \{Ax : A \in K\} \).

Let \( S \) be a subset of \( Y \). We denote by \( [S] \) the order-convex cover of \( S \), i.e.,

\[
[S] = \{y \in Y : a \leq y \leq b \text{ for some } a, b \in S\}.
\]

\( S \) is said to be order-convex if and only if \( S = [S] \).

**Definition** Let \( K \) be a subset of \( \mathcal{L}(X, Y) \). The hereditary order-convex cover of \( K \) is defined by

\[
[K]^h = \{A \in \mathcal{L} : Ax \in [Kx] \text{ for all } x \in X\}.
\]