AN ADDITIVE PROBLEM ABOUT POWERS OF FIXED INTEGERS

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1. Introduction.

Let $A$ be a set of distinct integers $\geq 2$ and $s$ a nonnegative integer. Define

$$\Sigma(\text{Pow}(A; s)) = \left\{ \sum_{a \in A, k \geq s} \varepsilon_{a,k} a^k, \varepsilon_{a,k} \in \{0, 1\} \right\}.$$

Burr, Erdős, Graham and Wen-Ching Li [1] proved several results providing sufficient conditions in order that $\Sigma(\text{Pow}(A; s))$ contains all sufficiently large integers. They also stated the following

CONJECTURE 1. Let $s \geq 1$. The set $\Sigma(\text{Pow}(A; s))$ contains all sufficiently large integers if and only if

$$\sum_{a \in A} \frac{1}{a - 1} \geq 1 \quad \text{and} \quad \gcd\{a \in A\} = 1.$$

In particular the conjectured result is independent of $s \geq 1$. In this paper we fix our attention for finite sets $A$. In section 2 we prove some
structural and density properties of \( \Sigma(\text{Pow}(A; s)) \). According to [3], we define the lower and upper asymptotic density of a subset \( M \) of naturals as 
\[
\liminf_{x \to \infty} \frac{M(x)}{x} \quad \text{and} \quad \limsup_{x \to \infty} \frac{M(x)}{x}
\]
respectively, where \( M(x) \) is the counting function of \( M \).

In [1] the following question is raised: what can we say about lower and upper asymptotic density of \( \Sigma(\text{Pow}(A; s)) \) when \( A \) is finite and

\[
\sum_{a \in A} \frac{1}{\log a} > \frac{1}{\log 2} \quad ?
\]

In Proposition 3 we show that, apart from the case \( A = \{2\} \), the relation (1) is a necessary condition in order that \( \Sigma(\text{Pow}(A; s)) \) has positive upper asymptotic density. We also provide an example which shows that the same condition is not sufficient.

We shall denote \( \Sigma(\text{Pow}(A)) = \Sigma(\text{Pow}(A; 0)) \). Note that \( \Sigma(\text{Pow}(A)) \) is trivial for infinite sets \( A \). A positive integer \( n \) will be called expressible for \( A \) if \( n \in \Sigma(\text{Pow}(A)) \). In case of nonambiguity we omit to precise "for \( A \)". Denote

\[
\begin{align*}
\text{P}_A(x) &= \#\{n \in \Sigma(\text{Pow}(A)), n < x\} \\
\text{P}_{A,s}(x) &= \#\{n \in \Sigma(\text{Pow}(A; s)), n < x\}
\end{align*}
\]

the counting function of \( \Sigma(\text{Pow}(A)) \) and \( \Sigma(\text{Pow}(A; s)) \) respectively. Erdős [2] asked for a proof that \( \Sigma(\text{Pow}([3, 4]; 1)) \) has positive lower asymptotic density. In [1] the same question is raised. Among other things, in this paper we prove that \( \text{P}_{[3,4]}(x) \gg x^{0.9659} \).

2. Structure properties.

The following proposition allows us to reduce the study of density properties of \( \Sigma(\text{Pow}(A; s)) \) to the case of \( \Sigma(\text{Pow}(A)) \).

**Proposition 1.** Let \( A = \{a_1, \ldots, a_k\} \) be a finite set of distinct positive integers \( \geq 2 \) and let \( s \) be a positive integer. Then \( P_A(x) \leq 2^s P_{A,s}(x) \). In particular \( \Sigma(\text{Pow}(A)) \) has positive lower \{upper\} asymptotic density if and only if \( \Sigma(\text{Pow}(A; s)) \) has positive lower \{upper\} asymptotic density.

**Proof.** For integers \( 1 \leq i \leq k \) and \( 0 \leq j \leq s - 1 \) fix \( E = \{\epsilon_{i,j}\}_{i=1,\ldots,k \atop j=0,\ldots,s-1} \) with \( \epsilon_{i,j} \in \{0, 1\} \) and define \( \psi_E(n) = n + \sum_{i,j} \epsilon_{i,j} a_i^j \). There are \( 2^s \)