HIGHER ORDER SECANT VARIETIES
AND BLOWN UP VARIETIES

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Let $\pi : X \to Y$ be the blowing up of the projective variety $Y$ at $s$ general points. Here we study the higher order secant varieties of the linearly normal embeddings of $X$ and $Y$ into projective spaces. We give conditions on the embedding of $Y$ which imply that the first $t$ secant varieties of a related embedding of $X$ have the expected dimension.

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In the last 15 years several authors studied (with very different motivations, aims and tools) the geometry associated to the following situation. Let $\pi : X \to Y$ be the blowing up of the projective variety at $s$ points $P_1, \ldots, P_s$. Many of these authors (see for instance [AH], [C], [GG], [GGP], [C] and the references there) were interested in the embeddings of $X$ into a projective space. Here we are interested in the case in which the $s$ points $P_1, \ldots, P_s$ of $Y$ are general points of $Y$. Here we will study the dimensions of the higher order secant varieties of such embeddings. These results have nice geometric consequences to the dimension and non emptiness of the set of 0-dimensional subschemes of $A$ and $B$ which, under general projections of $A$ (resp. $B$) into lower dimensional projective spaces spans a linear space of dimension at most their length - 2.
(see section 3). One of our source of inspiration for the search of these geometric results was [R] and in particular the discussion in [R] between Corollary 1.2 and Corollary 1.3.

We work always in characteristic 0. For background on higher order secant varieties, see [A1], [A2] and/or [A3]. Here is our first main result.

**THEOREM 0.1.** Let $Y$ be an integral complete variety, $\pi : X \to Y$ the blowing up of $Y$ at $s$ general points $P_i$, $1 \leq i \leq s$ and $M \in \text{Pic}(Y)$, $M$ very ample. Let $E_i := \pi^{-1}(P_i)$, $1 \leq i \leq s$, be the exceptional divisors. Set $L := \pi^*(M) - \sum_{1 \leq i \leq s} E_i \in \text{Pic}(X)$. Assume $L$ very ample. Let $h : Y \to \mathbb{P}$ (resp. $f : X \to \mathbb{P}$) be the map of $Y$ (resp. $X$) into a projective space induced by $M$ (resp. $L$). Set $A := h(Y)$, $B := f(X)$ and $n := \dim(Y)$. Assume $2s \geq h^0(Y, M) + 2$ and $s > n$. Let $k > 0$ be the first integer $t$ such that the higher order secant variety $B^{[t]}$ of $B$ has dimension less than the expected one, i.e. $\dim(B^{[t]}) < t(n + 1) - 1$. Then $B^{[k+n-1]} = \mathbb{P}$.

The proof of Theorem 0.1 will be given in section 1. See Remark 1.3 for a mild generalization of the statement of 0.1. Then (with the notations of the statement of 0.1) we study the projective geometry of the union of the embedded exceptional divisors $f(E_i)$ (see Proposition 1.6).

In section two we prove the following theorem.

**THEOREM 0.2.** Fix integers $n \geq 2$, $s \geq 1$ and $a_i$, $1 \leq i \leq s$, with $a_i > 0$ for every $i$. Let $Y$ be a complete $n$-dimensional variety and $M \in \text{Pic}(Y)$, $M$ an ample and spanned line bundle. Let $\pi : X \to Y$ be the blowing up of $Y$ at $s$ general points and $E_i$, $1 \leq i \leq s$, the exceptional divisors. Assume that $L := \pi^*(M) - \sum_{1 \leq i \leq s} a_i E_i \in \text{Pic}(X)$ induces (after deleting base divisors) a rational map on $X$ into a projective space $\mathbb{P}$ with image $B$ of dimension $n$. Assume $h^0(X, L) = h^0(Y, M) - \sum_{1 \leq i \leq s} (a_i + n - 1)!/(a_i!n!)$. Let $A$ be the image of $Y$ into a projective space $\mathbb{P}$ with respect to the rational map induced by the complete linear system $|M|$. Then for all integers $t$