DERIVATIONS IN PRIME RINGS

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Let $R$ be a prime ring and $D$ a nonzero derivation of $R$. If one of the four conditions holds in $R$, then $R$ is commutative:

(i) $X^2 D(X) - D(X)X^2 \in Z(R)$, $\text{Char } R \neq 2$;
(ii) $X^2 D(X) - XD(X)X \in Z(R)$, $\text{Char } R \neq 2$;
(iii) $X^3 D(X) - D(X)X^3 = 0$, $\text{Char } R \neq 2, 3$;
(iv) $X^m D(X) + X^m D(X)X^{n-m} \in Z(R)$, where $m, n$ are fixed integers, $0 < m < n$ and $\text{Char } R = 0$ or $\text{Char } R > n$.

A number of authors have generalized Posner's second theorem [1] in several ways (see [2], [3], and [4]). In this paper, we give some conditions on commutativity of prime rings with a nonzero derivation. Our methods are somewhat different from those employed by other authors. Throughout this paper, $R$ denote an associative ring with center $Z(R)$, and $D$ a nonzero derivation of $R$.

We shall need the following three lemmas.

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LEMMA 1. Let $R$ be a prime ring with $\text{Char} R = 0$ or $\text{Char} R > n$, $m$ and $n$ be fixed integers with $0 \leq m < n$. If $R$ satisfies either

(i) $X^n D(X) + X^m D(X)X^{n-m} \in \mathcal{Z}(R)$, or

(ii) $X^n D(X) - X^m D(X)X^{n-m} \in \mathcal{Z}(R)$

then $D(a) = 0$ for each $a$ in $R$ such that $a^2 = 0$.

Proof. If $R$ satisfies (i), by replacing $X$ by $ra$ in (I), we have

(1) $(ra)^n D(ra) + (ra)^m D(ra)(ra)^{n-m} \in \mathcal{Z}(R)$

By commuting (1) with $a$, we get

(2) $a((ra)^n D(ra) + (ra)^m D(ra)(ra)^{n-m}) = (ra)^n D(ra)a$

In (2), by multiplying $a$ from the left, we obtain

$a(ra)^n D(ra) a = 0$

that is

$a(ra)^{n-1} D(a) a = 0$.

Since $R$ is prime and $\text{Char} R = 0$ or $\text{Char} R > n$, then $D(a)a = 0$ by [5]. And $D(a^2) = aD(a) + D(a)a = 0$, so we get

(3) $aD(a) = D(a)a = 0$.

For all $r \in R$, by (3) we gain

(4) $aD(ar) = D(ra)a = 0$.

By induction, it is easy to see

$(ar + ara)^k = (ar)^k + (ar)^k a$.

So replacing $X$ by $ar + ara$ in (I), we obtain

(5) $(ar)^n D(ar) + (ar)^{n+1} D(a) + (ar)^n D(ar)a + (ar)^m D(ar)\cdot$

$(ar)^{n-m} + (ar)^m D(ar)(ar)^{n-m} a \in \mathcal{Z}(R)$.