MAPPINGS OF DEPENDENCE SPACES

by E. Lois and M. Benedicty (Pittsburgh, U. S. A.)

INTRODUCTION. The purpose of this paper is to study those mappings between $D$-spaces that preserve some amount of the structure of the spaces.

A $D$-space is the association $(S, I)$ of a non-empty set $S$ and a collection $I$ of subsets of $S$ such that the axioms $D1$, $D2$, $D3$ below are satisfied (see BENEDICTY [1]; compare VAN DER WAERDEN [6], JACOBSON [4], HIGGS [2], where different names - or no names at all - are used). The sets in $I$ are the independent sets; all the other subsets of $S$, as well as the families consisting of elements of $S$ and containing repetitions, are dependent. Such families need to be considered, although no particular sophistication is needed in dealing with them; those without repetitions are obviously identified with the corresponding subset of $S$. The empty set is $\emptyset$; it is finite by convention. The null set of $(S, I)$ is the set $N$ of the elements $z$ of $S$ such that $|z| \notin I$.

$D1$. If $A \in I$ and $B \subseteq A$, then $B \in I$.

$D2$. If $A \in I$, $B \in I$, $A = \{a_1, \ldots, a_r\}$, $B = \{b_1, \ldots, b_s\}$, $x \in S$, each family $\langle A, b \rangle$ (short for $\langle a_1, \ldots, a_r, b \rangle$) is dependent, and the family $\langle B, x \rangle$ is dependent, then the family $\langle A, x \rangle$ is dependent.

$D3$. $\emptyset \in I$.

The element $x$ depends on the subset $X$ if there exists a finite independent subset $A$ of $X$ such that $\langle A, x \rangle$ is dependent. Also, $x$ depends on the family $B$ if it depends on the set of the elements that appear in $B$. If $X$ is a family,
then its closure \([X]\) is the set of the elements that depend on \(X\); it is clearly the same as the closure of the set of the elements that appear in \(X\). The closed sets are those that coincide with their own closure. The rank of \(X\) is \(\infty\) or, when existent, the maximum number of elements of the finite independent sets consisting of elements that appear in \(X\).

The following properties are valid in a \(D\)-space (see [1]): P1. if \(X \subseteq S\), then \(X \subseteq [X] = [[X]]\); P2. if \(X \subseteq [Y]\) (in particular, if \(X \subseteq Y\)), then \([X] \subseteq [Y]\); P3. if \(A = \{a_1, \ldots, a_r\} \subseteq I, r > 0, b \in [A], b \notin [a_1, \ldots, a_{r-1}]\), then \(a_r \in [[a_1, \ldots, a_{r-1}, b]]\).

They imply that the association of \(S\) and the operation of closure satisfies Axioms (i) (ii) (iii) of HIGGS [2]. Suppose that \((S, I)\) satisfies also: D5. if \(x \in S\), then \([x] \in I\); and D6. if \(x \in S, y \in S, x \neq y\), then \([x, y] \in I\). Then \((S, [\cdot])\) satisfies also Axiom (iv) of [2] and is therefore an incidence space. The independent sets of \((S, I)\) are not necessarily the same as those defined by HIGGS [2]; the finite ones are the same; the infinite ones according to HIGGS are the amplest collection that gives rise to the same closure as defined in this paper; this is the collection of all those infinite sets that do not include any finite dependent set.

**MAPPINGS.** Let \((S, I), (S^*, I^*)\) be \(D\)-spaces and \(\omega: S \rightarrow S^*\) a mapping.

**Definitions.** (1) The mapping \(\omega\) is graphic if \(\omega[X] \subseteq [\omega X]\) for every subset \(X\) of \(S\); (2) it is a linearity if \(\omega[\langle y_0, y_1 \rangle] \subseteq [\omega \langle y_0, y_1 \rangle]\) whenever \(y_0, y_1\) are in \(S\); (3) it is a strong homomorphism if the family \(\omega \langle X \rangle\) is dependent in \((S, I)\) whenever \(X\) is a dependent family in \((S, I)\) \(\omega \langle X \rangle\) is defined as the family whose members are \(\omega x\), when \(x\) runs through the elements of the family \(X\) and the indexing is preserved or altered in an equivalent fashion); (4) it is a weak homomorphism if \(\omega \langle X \rangle\) is dependent whenever \(X\) is a finite dependent family in \((S, I)\). (5) (6) The mapping \(\omega\) is a linear mapping (weak or strong) if it is both a linearity and a homomorphism.

**Remark 1.** Later on we shall consider mappings for which \(\omega^{-1}\) (although not a mapping and although defined only on \(\omega S\)) satisfies analogous conditions, such as \(\omega^{-1}[X^*] \subseteq [\omega^{-1} X^*]\), \(\omega^{-1}[\langle x_0^*, y_1^* \rangle] \subseteq [\omega^{-1} \langle x_0^*, y_1^* \rangle]\). Incidentally, all of the arguments in this paper can easily be adapted to the case when \(\Omega\) is a relation between \(S\) and \(S^*\) (i.e. \(\Omega \subseteq S \times S^*\)) and \(\omega X(\omega^{-1} X^*)\) is defined as \([x^*] \therefore \exists x \in X \therefore (x, x^*) \in \Omega \cap (X \times S^*)\) \((|x| \therefore \exists x^* \therefore (x, x^*) \in \Omega \cap (S \times X^*))\).

**Remark 2.** It is evident that: every graphic mapping is a linearity; a weak homomorphism is a mapping that preserves dependence of the families having finite rank; every strong homomorphism is a weak homomorphism.