When is a hidden variable theory compatible with quantum mechanics?

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Abstract. This paper is devoted to a study of some of the basic conditions which have to be satisfied by a hidden variable theory in order that it can reproduce the quantum mechanical probabilities. Of course one such condition, which emerges from the important theorem of Bell, is that a hidden variable theory has to be non-local. It is shown that a hidden variable theory is also incompatible with the conventional interpretation of mixed states and the mixing operation in quantum theory. It is therefore concluded that, apart from being non-local, a hidden variable theory would also necessarily violate the usual assumption of quantum theory that the density operator provides an adequate characterization of any ensemble of systems, pure or mixed.

Keywords. Hidden variable theories; complete specification of the state of a system; compatibility with quantum mechanics; local causality; density operators; mixed ensembles; quantum mechanics.

1. Introduction: The framework of hidden variable theories

In conventional non-relativistic quantum theory the state of a system is represented by a density operator \( \rho \) on a Hilbert space \( \mathcal{H} \). A measurement performed on the system (normally considered to be instantaneous) is characterised by a self-adjoint operator \( A \) on \( \mathcal{H} \), such that the probability that the result of the measurement lies in a Borel set \( \Delta \subset \mathbb{R} \), is given by

\[
\Pr_A^\rho (\Delta) = \text{Tr} (\rho \, P_A(\Delta)),
\]

where \( \Delta \rightarrow P_A(\Delta) \) is the spectral measure associated with the self-adjoint operator \( A \). Usually, equation (1) is interpreted as implying that if a large number of systems (i.e. an ensemble) is prepared according to a procedure characteristic of the ‘quantum state’ \( \rho \), and if each of them is subjected to a measurement characterised by the ‘observable’ \( A \), then \( \Pr_A^\rho (\Delta) \) is the fraction of systems which yield an outcome in the set \( \Delta \subset \mathbb{R} \).

For various motivating reasons, it has been argued (ever since the inception of quantum theory) that the above specification of the state of a system in quantum theory should be ‘completed’, so that the completed theory would predict the actual outcomes of experiments performed on individual systems, and not merely the statistics of the outcomes of experiments performed on ensembles. Such a ‘completion’ is sought to be achieved by constructing a ‘hidden variable theory’ where one in-
cludes in the characterization of the state of a system, certain additional variables or parameters, (which are 'hidden' as far as quantum theory is concerned), over and above the quantum-theoretic state of the system. The general framework of such hidden variable theories is the following (For general reviews of the subject, we refer to Belinfante 1973; Bell 1966, 1971a; Bohm and Bub 1966; Bub 1974; Capasso et al 1970; Clauser and Shimomy 1978; d'Espagnat 1976, 1979; Fine 1976; Kochen and Specker 1967; Jammer 1974; Roy 1980; Selleri and Tarozzi 1981; Vandana Shiva 1978; Virendra Singh 1980 and Wigner 1980).

Associated with each quantum system is a measure space $(\Gamma, \mathcal{B})$ such that the following conditions are satisfied:

(I) With each density operator $\rho$ is associated a probability measure $\mu^\rho$ on $(\Gamma, \mathcal{B})$; with each self-adjoint operator $A$ is associated a real valued random variable $X_A: \Gamma \to \mathbb{R}$. 
(II) For each state $\rho$ and observable $A$,

$$
\Pr^\rho_A (\Delta) = \text{Tr} \left[ \rho P^A (\Delta) \right] = \mu^\rho \left[ X_A^{-1} (\Delta) \right],
$$

for all Borel sets $\Delta \subseteq \mathbb{R}$.

The interpretation of the above framework is the following. $\Gamma$ is the set of all 'completely-specified states' (or the so-called 'dispersion-free states') which are now characteristic of individual systems; in other words, each point $\lambda \in \Gamma$ is thought of as providing a 'complete-specification' of the state of an individual system. In the general framework discussed in this paper, no further assumptions are made regarding the space $\Gamma$, though in many models it turns out that each point $\lambda \in \Gamma$ is specified by a collection of variables $(\rho, \theta_1, \theta_2, \ldots)$ where $\rho$ is a density operator (i.e. the quantum-theoretic state of the system) and $\theta_1, \theta_2, \ldots$ are the additional (or 'hidden') variables needed to complete the quantum-theoretic description. Now, if an individual system is in the completely-specified state $\lambda \in \Gamma$, then the value of an observable $A$, (that one obtains in a measurement of $A$), is the number $X_A (\lambda)$. Also, if an ensemble of systems is prepared according to a procedure characteristic of the quantum state $\rho$, then associated with this ensemble is a probability measure $\mu^\rho$ on $\Gamma$, which describes the way in which the various individual systems of the ensemble are distributed over the points of $\Gamma$. The requirement (II) is precisely the condition that the individual systems of the ensemble are distributed over the points of $\Gamma$ in such a way that the statistical distribution of experimental outcomes is exactly the same as that predicted by quantum theory.

Apart from (I) and (II), another requirement that we need to impose on a hidden variable theory is the following spectrum rule (see Fine 1976):

(III) For each observable $A$,

$$
X_A (\lambda) \in \sigma (A);
$$

where $\sigma (A)$ is the spectrum of $A$.

This requirement follows from the generally accepted principle of quantum theory that in any experiment to measure $A$, the outcome is always a number belonging to the spectrum of $A$.