Fourier transform of single eigenvalue probability density function using ensemble-averaged traces of the Hamiltonian

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Abstract. A determinantal identity is used to calculate the ensemble-averaged traces of the Hamiltonian. Using these averages a general expression is obtained for the Fourier transform of the single eigenvalue probability density function for all the three Gaussian ensembles for the two-dimensional case. It is shown how one can use the familiar step-up operators for the representation of a determinant. The ensemble-averaged traces are also used to derive the Fourier transform of the non-zero mean ensemble.

Keywords. Matrix ensemble theory; Fourier transform; Grassmann variables.

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1. Introduction

Since the introduction of matrix ensemble theory by Wigner (Mehta 1967), one of the challenging problems has been to study the probability density function of the single eigenvalue. The method which has been commonly followed is to transform the distribution of the Hamiltonian matrix elements to the joint distribution of the eigenvalues and eigenvector components and formally integrate over the eigenvectors to get the distribution of the eigenvalues of the Hamiltonian. Using this distribution one integrates over all the eigenvalues but one, employing special techniques, if needed, to obtain the probability density function of the single eigenvalue.

Recently (Verbaarschot et al. 1984; Ullah 1981) it has been found to be advantageous to employ directly the Hamiltonian matrix element distribution and the invariant relations between the eigenvalues and the traces of the Hamiltonian matrix. This technique is found to be particularly useful if the ensemble is non-Gaussian. In a recent study of Gaussian unitary ensemble (Ullah 1985), it was found that the Fourier transform of the single eigenvalue probability density function has an extremely simple form. This has prompted us to study the Fourier transform of the single eigenvalue probability density function for all the three Gaussian ensembles. In the present work we shall study matrix ensembles of low dimensions only. In obtaining the Fourier transform we shall make use of the ensemble-averaged traces. The ensemble-averaged traces will be obtained using a determinantal identity in §2. The transforms for $N = 2$ will be given in §3. In §4 we shall discuss the use of step-up angular momentum operators in place of Grassmann variables (Balian and Zinn-Justin 1975; Efetov 1982) in finding the ensemble average of a determinant. The technique developed in the present
manuscript will also be applied to non-zero mean matrix ensembles (Edwards and Jones 1976) having \( N = 2 \) in §5 and the conclusion in §6.

2. Ensemble-averaged traces of the Hamiltonian

Let us consider a 2 x 2 Gaussian orthogonal ensemble (GOE) in which each diagonal Hamiltonian matrix element has variance \( \frac{1}{2} \) and the off-diagonal element has variance \( \frac{1}{4} \). We now use the well-known determinantal identity

\[
\det (1 - \lambda H) = \exp \left( - \sum_{k=1}^{\infty} \frac{\lambda^k}{k} \langle \text{Tr} H^k \rangle \right),
\]

(1)

to calculate the ensemble-averaged traces. The identity can be easily seen to be true by going over to the eigenvalues of \( H \). Taking log, we can write the ensemble average as

\[
- \sum_{k=1}^{\infty} \frac{\lambda^k}{k} \langle \text{Tr} H^k \rangle = \left\{ \int \exp \left( H_{11}^2 + H_{22}^2 + 2H_{12}^2 \right) \prod_{i < j} \text{d}H_{ij} \right\}^{-1}
\]

\[
\left\{ \int \ln \left[ 1 - \lambda (H_{11} + H_{22}) + \lambda^2 (H_{11}H_{22} - H_{12}^2) \right] \exp - (H_{11}^2 + H_{22}^2 + 2H_{12}^2) \prod_{i < j} \text{d}H_{ij} \right\}. \quad (2)
\]

Using first the transformation \( H_{11} + H_{22} = u, \ H_{11} - H_{22} = v \) and then putting \( v = \rho \cos \theta \) and \( 2H_{12} = \rho \sin \theta \), we get after some simplification

\[
\langle \text{Tr} H^k \rangle = 2^{-k} \left\{ \int_{\rho=0}^{\infty} \int_{u=-\infty}^{\infty} \left[ \exp - \frac{1}{2}(u^2 + \rho^2) \right] \rho \text{d}\rho \text{d}u \right\}^{-1}
\]

\[
\left\{ \int_{\rho=0}^{\infty} \int_{u=-\infty}^{\infty} \left[ \exp - \frac{1}{2}(u^2 - \rho^2) \right] [(u + \rho)^k + (u - \rho)^k] \right\}. \quad (3)
\]

It is obvious from expression (3) that if \( k \) is odd then the ensemble-averaged trace vanishes. For \( k = 2m \), one can carry out the integrations in (3) and obtain the following expression for \( \langle \text{Tr} H^{2m} \rangle \),

\[
\langle \text{Tr} H^{2m} \rangle = \frac{\Gamma(m + \frac{1}{2})}{\sqrt{\pi 2^{-m-1}}} F(-m, 1; \frac{1}{2}; -1), \quad (4)
\]

where \( F \) denotes the hypergeometric function (Abramowitz and Stegun 1965).

The same procedure can be used for the Gaussian unitary ensemble (GUE). If we again take the same variances for diagonal and the real and imaginary parts of the off-diagonal element, then we get the following expression for \( \langle \text{Tr} H^{2m} \rangle \),

\[
\langle \text{Tr} H^{2m} \rangle = \frac{\Gamma(m + \frac{1}{2})}{\sqrt{\pi 2^{-m-1}}} F(-m, \frac{3}{2}; \frac{1}{2}; -1). \quad (5)
\]