Painlevé analysis and exact solutions of two dimensional Korteweg–de Vries–Burgers equation

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Abstract. Two dimensional Korteweg–de Vries–Burgers equation is shown to be non-integrable using Painlevé analysis. Exact travelling wave solutions are obtained using an algorithmic approach of truncating the Painlevé series expansions.

Keywords. Korteweg–de Vries–Burgers equation; Painlevé analysis; exact solutions.

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1. Introduction

Painlevé analysis is a powerful tool in investigating the integrability properties of differential equations [1, 2, 3]. It can be used to find Lax pairs, Bäcklund transformations, Hirota’s bilinear equations, symmetries, invariants, etc., of integrable equations. It can also be used to find exact solutions of non-integrable equations in an algorithmic way [4]. In this paper we present the Painlevé analysis of two dimensional Korteweg–de Vries–Burgers equation (2d-KdVB),

\[(u_t + uu_x + \mu u_{xxx} - \nu u_{xx})_x + \sigma u_{yy} = 0.\]  \hspace{1cm} (1)

It is a two-dimensional generalization of KdVB equation which is used as a non-linear wave model of fluid flow in an elastic tube with dispersion and dissipation, flow of liquids containing small bubbles, etc. [5, 6]. The 2d-KdVB serves as a model for propagation of shallow water waves subject to a small transverse disturbance and influenced by viscosity. Similarity solutions of such systems are discussed in [7]. Recently travelling wave solutions of this system were derived using different methods [8, 9, 10]. In all these methods they assumed solutions with travelling wave form and substituted such a solution in the system and determined the exact parameter values at which they exist, if they exist.

Using the technique of truncating the Painlevé series expansions at different orders we obtained exact travelling wave solutions of this equation without assuming any particular form for the solutions a priori. Equation (1) is found to be non-Painlevé type and due to Painlevé conjecture it is non-integrable, but it has got conditional Painlevé property. When \( \nu = 0 \) and \( \sigma = 0 \) it becomes KdV equation and when \( \nu = 0 \) it becomes KP equation. Both of them are integrable and have soliton solutions. When \( \mu = 0 \) and \( \sigma = 0 \) it becomes Burgers equation which is also integrable. At \( \sigma = 0 \) it is KdVB
equation and at $\mu = 0$ it is 2d-Burgers equation, both of them are non-integrable. Painlevé analysis of KdVB is given in [11].

In the next section we present the Painlevé analysis of 2d-KdVB. In §3 exact solutions of the system are given. Last section summarises the results and conclusion.

2. Painlevé analysis

In Painlevé analysis we expand the solution $u$ about a singular manifold $\phi(x, y, t) = 0$ in an infinite series

$$u = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j,$$

(2)

where $\alpha$ is a negative integer determined by balancing the powers of $\phi$ of dominant terms in the equation. $\phi$ is a non-characteristic manifold. Coefficients $u_j$ are functions of $x, y$ and $t$. If solutions are single valued about the movable singular manifold, the partial differential equation is said to have Painlevé property. There are basically three steps in the Painlevé analysis, viz, dominant behaviour analysis, finding the resonances, and checking whether arbitrary coefficients enter at the resonance values [3].

From the dominant behaviour analysis we get $\alpha = -2$. By balancing the terms of order $\phi^{-6}$ in the equation after substituting (2) for $u(x, y, t)$ in (1), we obtain recurrence relations for $u_j$

$$(j - 4)(j - 5)u_{j-2} \phi_x \phi_t + (j - 5)[u_{j-3} \phi_{x t} + u_{j-3,x} \phi_t + u_{j-3,x} \phi_x]$$

$$+ u_{j-4,x} + (j - k - 2)(k - 2)u_{j-k} \phi_x^2 + 2(j - k - 2)u_{j-k} \phi_{x,k-1,x} \phi_x$$

$$+ u_{j-k-1,x} \phi_{k-1,x} + (k - 2)(k - 3)u_{j-k} \phi_x^2 + (k - 3)u_{j-k} \phi_{x,k-1} \phi_{xx}$$

$$+ 2(k - 3)u_{j-k} \phi_{k-1,x} \phi_x + u_{j-k} \phi_{k-2,xx} + \mu \{ (j - 2)(j - 3)(j - 4)(j - 5)u_j \phi_x^4$$

$$+ (j - 3)(j - 4)(j - 5)[6u_{j-1} \phi_x^2 \phi_{xx} + 4u_{j-1,x} \phi_x^3]$$

$$+ (j - 4)(j - 5)[3u_{j-2} \phi_x^2 + 4u_{j-2,x} \phi_x \phi_{xx}$$

$$+ 12u_{j-2,x,x} \phi_x^2 + 6u_{j-2,xx,xx} \phi_x^2] + (j - 5)[u_{j-3} \phi_{xxxx} + 4u_{j-3,x} \phi_{xxx}$$

$$+ 6u_{j-3,xx} \phi_x + 4u_{j-3,xxx} \phi_x] + u_{j-4,xxxx}\}

$$- \nu \{ (j - 4)(j - 5)[(j - 3)u_{j-1} \phi_x^3 + 3u_{j-2} \phi_x^2 \phi_{xx} + 3u_{j-2,x} \phi_x^2]$$

$$+ (j - 5)[u_{j-3} \phi_{xxx} + 3u_{j-3,x} \phi_{xx} + 3u_{j-3,xx} \phi_x] + u_{j-4,xxx}\}$$

$$+ \sigma \{ (j - 5)[(j - 4)u_{j-2} \phi_x^2 + u_{j-3} \phi_{yy} + 2u_{j-3,x} \phi_y] + u_{j-4,yy}\} = 0,$$

(3)

where coefficients $u_j$ with negative $j$ are taken to be zero. For $j = 0$ we obtain

$$u_0 = -12\mu \phi_x^2.$$

(4)

Using (4) in (3) collecting coefficients of $u_j$ we obtain

$$(j + 1)(j - 4)(j - 5)(j - 6)u_j = h_j(\phi_x, \phi_y, \phi_1, \ldots, u_0, \ldots, u_{j-1}),$$

(5)

where $h_j$ is a non-linear function. We can see that $j = -1, 4, 5, 6$, are resonances at which $u_j$ becomes arbitrary. Resonance at $-1$ corresponds to the arbitrariness of $\phi$. 

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