A STRUCTURE THEOREM FOR THE ABSOLUTE RIESZ SUMMABILITY
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1. DEFINITION. Let \( \sum a_n \) be a given infinite series and let \( \lambda_n = \lambda(n) \) be a positive monotonic function of \( n \), tending to infinity with \( n \). We write

\[
A_\lambda^\omega(\omega) = A_\lambda^0(\omega) = \sum_{\lambda_n \leq \omega} a_n,
\]

\[
A_\lambda^\omega(\omega) = \sum_{\lambda_n \leq \omega} (\omega - \lambda_n)^r a_n, \quad r > 0.
\]

The series \( \sum a_n \) is said to be summable \( (R, \lambda_n, r) \), \( r \geq 0 \), to sum \( s \), if \( A_\lambda^\omega(\omega)/\omega^r \to s \), as \( \omega \to \infty \), and is said to be absolutely summable \( (R, \lambda_n, r) \), or summable \( |R, \lambda_n, r| \), \( r \geq 0 \), if \( A_\lambda^\omega(\omega)/\omega^r \in BV(A, \infty) \) \(^{(1)}\), where \( A \) is a finite positive number \(^{(2)}\).

The sequence \( \{\lambda_n\} \) is called the "type" and the number \( r \) is called the "order".

An equivalent definition is obtained, as follows, by a suitable extension of the definition of the type \( \lambda(x) \) at points other than those given by \( x = n \) \((n = 1, 2, \ldots)\), and a corresponding change in the variable involved in the "Riesz mean" \( A_\lambda^\omega(\omega)/\omega^r \).

\(^{(1)}\) By "\( f(x) \in BV(h, k) \)" we mean that \( f(x) \) is a function of bounded variation over the interval \((h, k)\).

\(^{(2)}\) Obrechkoff [3], [4].
Let $\lambda = \lambda (\omega)$ be a continuous, differentiable and monotonic increasing function of $\omega$ in $(A, \infty)$, where $A$ is a positive constant and let $\lambda (\omega)$ tend to infinity with $\omega$. We write

$$
C_r(\omega) = \sum_{n \leq \omega} [\lambda (\omega) - \lambda (n)]^{r} a_n, \quad r > 0.
$$

Then $\sum a_n$ is said to be absolutely summable $|R, \lambda, r|$, $r \geq 0$, if the integral

$$
\int_{A}^{\infty} d \left| \frac{C_r(\omega)}{\left| \lambda (\omega) \right|^{r+1}} \right|
$$

converges.

Now, for $r > 0$, $m < \omega < m + 1$,

$$
\frac{d}{d \omega} \left| \frac{C_r(\omega)}{\left| \lambda (\omega) \right|^{r+1}} \right| = \frac{r \lambda'(\omega)}{\left| \lambda (\omega) \right|^{r+1}} \sum_{n \leq \omega} \left| \frac{\lambda (\omega) - \lambda (n)}{\lambda (\omega)} \right|^{r-1} \lambda (n) a_n.
$$

Hence, the summability $|R, \lambda, r|$, $r > 0$, is equivalent to the convergence of the integral

$$
\int_{A}^{\infty} \left| \frac{r \lambda'(\omega)}{\left| \lambda (\omega) \right|^{r+1}} \sum_{n \leq \omega} \left| \frac{\lambda (\omega) - \lambda (n)}{\lambda (\omega)} \right|^{r-1} \lambda (n) a_n \right| d \omega.
$$

Evidently, summability $|R, \lambda, 0|$ is equivalent to absolute convergence, whatever be the type $\lambda (\omega)$.

For convenience, we shall adopt here the alternative definition given above.

2. NOTATIONS. Let $f(t)$ be a periodic function with period $2\pi$ and integrable $(L)$, that is integrable in the sense of Lebesgue, over $(-\pi, \pi)$. We can, without any loss of generality, write the Fourier series of $f(t)$ as

$$
\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t),
$$

assuming that the constant term is zero. Then the conjugate series of the Fourier series of $f(t)$ is given by

$$
\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t).
$$