A CHARACTERIZATION OF H-CLOSED URYSOHN SPACES

by Larry L. Herrington and Paul E. Long (Fayetteville, U.S.A.)

SUMMARY. The graph \( G(f) \) of \( f: X \to Y \) is defined to be \( * \)-closed if for each \( (x, y) \notin G(f) \) there exist open sets \( U \) and \( V \) containing \( x \) and \( y \), respectively, such that \( U \times \text{Int}(\text{cl}(V)) = \emptyset \). A characterization of \( H \)-closed Urysohn spaces is obtained using functions with \( * \)-closed graphs.

Definition 1. Let \( f: X \to Y \) be any function. The graph \( G(f) \) of \( f \) is \( * \)-closed if for each \( (x, y) \notin G(f) \) there exist open sets \( U \) and \( V \) containing \( x \) and \( y \), respectively, such that \( U \times \text{Int}(\text{cl}(V)) = \emptyset \).

Lemma 1. The graph of \( f: X \to Y \) is \( * \)-closed if and only if for each \( x \in X \) and \( y \in Y \) such that \( y \neq f(x) \), there exists an open set \( U \) containing \( x \) and an open set \( V \) containing \( y \) such that \( f(U) \cap \text{Int}(\text{cl}(V)) = \emptyset \).

If \((Y, T)\) is a topological space, we denote by \( T_* \) the topology on \( Y \) generated by the collection of all regular-open sets in \((Y, T)\) as a base. The topology \( T_* \) is called the semi-regular topology induced by \( T \).

Theorem 1. The function \( f: X \to (Y, T) \) has a \( * \)-closed graph if and only if \( f: X \to (Y, T_*) \) has a closed graph.

Proof. For the set \( G(f) \subset X \times (Y, T) \) to be \( * \)-closed is precisely that \( G(f) \subset X \times (Y, T_*) \) be closed.

Of course, a function with a \( * \)-closed graph also has a closed graph. In view of Theorem 1, the concepts of \( * \)-closed graphs and closed graphs coincide for functions \( f: X \to Y \) where \( Y \) has the semi-regular topology. In general, however, a function with a closed graph need not have a \( * \)-closed graph as Example 1 shows.
Definition 2 [5]. A function \( f: X \to Y \) is almost-continuous if for each \( x \in X \) and open \( V \) containing \( f(x) \), there exists an open \( U \) containing \( x \) such that \( f(U) \subseteq \text{Int}(\text{cl}(V)) \).

Theorem 2. Let \( f: X \to (Y, T) \) be almost-continuous where \( (Y, T) \) is Hausdorff. Then \( f \) has a \( * \)-closed graph.

Proof. The space \( (Y, T) \) is Hausdorff if and only if \( (Y, T_\#) \) is Hausdorff. Also, \( f: X \to (Y, T) \) is almost-continuous if and only if \( f: X \to (Y, T_\#) \) is continuous. Therefore, \( f: X \to (Y, T_\#) \) has a closed graph [2, Theorem 1.5 (3), p. 140] so that \( f: X \to (Y, T) \) has a \( * \)-closed graph by Theorem 1.

Definition 3 [4]. A space \( Y \) is nearly-compact if and only if each regular-open cover of \( Y \) has a finite subcover.

Theorem 3. Let \( (Y, T) \) be a nearly-compact space. For every topological space \( X \), each \( f: X \to (Y, T) \) with a \( * \)-closed graph is almost-continuous.

Proof. Theorem 1 shows \( f: X \to (Y, T_\#) \) has a closed graph. Since \( (Y, T) \) is nearly-compact if and only if \( (Y, T_\#) \) is compact, \( f: X \to (Y, T_\#) \) is continuous. It follows that \( f: X \to (Y, T) \) is almost-continuous.

The following example shows the \( * \)-closed graph condition of Theorem 3 cannot be relaxed to a closed graph condition.

Example 1. Let \( Y = [0, 1] \times [0, 1] \) have the topology \( T \) generated by the usual open sets and sets of the form \( Y - (B \times [0]) \) where \( B \subseteq Q \cap [0, 1] \) (\( Q \) is the rationals) as a subbase. Note that \( Y \) is a nearly-compact Hausdorff space. Let \( A = Q \cap [0, 1] \) and define \( f: A \to (Y, T) \) by \( f(x) = (x, 0) \) if \( x \neq 0 \) and \( f(0) = (1, 1) \). Evidently \( f \) has a closed graph. However, since the semi-regular topology on \( Y \) is the usual topology, it is easy to see that \( f: X \to (Y, T) \) does not have a closed graph, hence \( f: X \to (Y, T_\#) \) does not have a \( * \)-closed graph. Also, \( f: X \to (Y, T_\#) \) is not continuous at \( x = 0 \) so that \( f: X \to (Y, T) \) is not almost-continuous.

We now let \( S \) be the class of topological spaces containing the class of Hausdorff, completely normal and fully normal spaces and use the results of Professor Kasahara [3] to obtain the following characterization of nearly-compact spaces. However, we first define a weakly Hausdorff space.

Definition 4 [6]. A space \( Y \) is weakly Hausdorff if for each \( y \in Y \), \( \{y\} = \bigcap \{F \subseteq Y : F \text{ is regular-closed in } Y \text{ and contains } y\} \).