IDENTITIES OF SYMMETRIC AND SKEW-SYMMETRIC MATRICES IN CHARACTERISTIC $p$

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It is well known how the Kostant-Rowen Theorem extends the validity of the famous Amitsur-Levitzki identity to skew-symmetric matrices. Here we give a general method, based on a graph theoretic approach, for deriving extensions of known permanental-type identities to skew-symmetric and symmetric matrices over a commutative ring of prime characteristic. Our main result has a typical Kostant-Rowen flavour: If $M \geq p[n + 1/2]$ then

$$C_M(X, Y) = \sum_{\alpha, \beta \in \text{Sym}(M)} x_{\alpha(1)}y_{\beta(1)}x_{\alpha(2)}y_{\beta(2)}\cdots x_{\alpha(M)}y_{\beta(M)} = 0$$

is an identity on $M_n^-(\Omega)$, the set of $n \times n$ skew-symmetric matrices over a commutative ring $\Omega$ with $p1_\Omega = 0$ (provided that $P > \sqrt{n + 1/2}$). Otherwise, the stronger condition $M \geq pn$ implies that $C_M(X, Y) = 0$ is an identity on the full matrix ring $M_n(\Omega)$.

1. Introduction.

The main aim of the present paper is to give a method for deriving identities of permanental type for symmetric and skew-symmetric matrices over commutative rings of prime characteristic.

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Our method combines, ingenious ideas of Rosset [5] and of Rowen [6] in order to count modulo $p$ the number of certain Eulerian paths in directed or partially directed graphs. While Rosset's proof of the Amitsur-Levitzki Theorem uses the exterior algebra, we shall replace it by a similar, but commutative algebra: we shall consider incidence matrices of graphs over the factor algebra of the commutative polynomial algebra $Q[v_1, v_2, ...]$ with respect to the ideal generated by the monomials $v_1^2, v_2^2, ...$

The most important tool we use is the following observation due to Rowen: for skew-symmetric $2m \times 2m$ matrices $U, V$ over a field of characteristic zero we have

\begin{equation}
(UV)^m - \mu_1(UV)^{m-1} + \mu_2(UV)^{m-2} - \ldots + (-1)^m \mu_m(UV)^0 = 0,
\end{equation}

where

\begin{equation}
\mu_0 = 1, \quad 2k\mu_k = \sum_{i=1}^{k} (-1)^{i-1} \mu_{k-i} tr(UV)^i, \quad 1 \leq k \leq m.
\end{equation}

We note that the above identities have been also fruitfully exploited in recent papers of Ma Wenxin-Racine [4] and Giambruno-Valenti [2].

Making use of the graph-theoretical results of Section 2, the first theorem we prove in Section 3 concerns the double permanental Capelli polynomial

\begin{equation}
C_M(X, Y) = \sum_{\alpha, \beta \in \text{Sym}(M)} x_{\alpha(1)} y_{\beta(1)} x_{\alpha(2)} y_{\beta(2)} \ldots x_{\alpha(M)} y_{\beta(M)}
\end{equation}

in the set \{x_1, x_2, \ldots, x_M, y_1, y_2, \ldots, y_M\} of non-commuting indeterminates.

It was proved in [3] that $C_M(X, Y) = 0$ is an identity of $M_n(\Omega)$ if and only if $M \geq pn$, where the prime $p \geq 2$ is the characteristic of the commutative ring $\Omega$.

For $p > \sqrt{n + 1/2}$, $p \neq 2$ and for skew-symmetric $x$'s and $y$'s we shall prove the stronger assertion, that $C_M(X, Y) = 0$ is an identity in $M_n(\Omega)$ if $M \geq p[n + 1/2]$.

Apart from the curious condition $p > \sqrt{n + 1/2}$ this is reminiscent to Giambruno's result on ordinary double Capelli