A Variational Property for a Family of Vector-Valued Functions (*)

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§ 1. - Let \( X \) be a Banach space and let \( A \) generate a strongly continuous group \( \{G(t): t \in \mathbb{R} = (-\infty, \infty)\} \) of uniformly bounded linear operators on \( X \). Let \( k: \mathbb{R} \to X \) be strongly continuously differentiable and let \( u(t, x) \) be the solution at time \( t \) of the Cauchy problem

\[
\frac{du(t)}{dt} = Au(t) + k(t) \quad (t \in \mathbb{R}), \quad u(0) = x.
\]

For \( x \) in the domain of \( A \), \( u \) exists and is given by

\[
u(t, x) = G(t)x + \int_0^t G(t-s)k(s) \, ds = G(t)x + h(t)
\]

for \( t \in \mathbb{R} \). Since \( A \) and \( k \) are given it follows that \( G \) and \( h \) are known. For arbitrary \( x \in X \) we may regard \( (2) \) as defining the unique generalized solution of \( (1) \). We regard the initial data \( x \) as a control variable and we consider the cost function

\[
\mu(x) = \sup \{ \|u(t, x)\| : t \in \mathbb{R}\}.
\]

The problem is to minimize the cost. It follows from the theorem given below that there exists a unique choice of \( x \) which minimizes the cost if \( X \) is uniformly convex, \( h \) is bounded, and the cost is finite for some choice of initial data.

§ 2. - We shall consider \( u(t, x) \) in a much more general framework than that described by \( (2) \). Let \( \mathcal{B}(X) \) denote the space of all bounded linear oper-

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ators on $X$. Let $G: \mathbb{R} \rightarrow \mathcal{B}(X)$ and let $h: \mathbb{R} \rightarrow X$ be arbitrary functions, and set

$$u(t, x) = G(t)x + h(t)$$

for $x \in X$ and $t \in \mathbb{R}$. Define

$$\mu(x) = \sup \{ \|u(t, x)\| : t \in \mathbb{R} \}$$

and call $\{x_n\}_1^{\infty}$ a minimizing sequence if

$$\lim_{n \rightarrow \infty} \mu(x_n) = \inf \{ \mu(x) : x \in X \}.$$  

**Theorem.** Let $X$ be uniformly convex. Let $\mathbb{R} \rightarrow X$, $G: \mathbb{R} \rightarrow \mathcal{B}(X)$, and assume the following three conditions.

(A1) For each $T > 0$, $\sup \{ \|G(t)\| : |t| < T \} < \infty$.

(A2) There is an $\varepsilon > 0$ such that $\|G(t)x\| \leq \varepsilon \|x\|$ for each $x \in X$ and each $t \in \mathbb{R}$.

(A3) $\{x \in X : \mu(x) < \infty\} \neq \emptyset$.

Let $\{x_n\}_1^{\infty}$ be a minimizing sequence. Then $\{u(t, x_n)\}_n^{\infty}$ is uniformly convergent (in the norm topology of $X$) for $t$ in compact subsets of $\mathbb{R}$. Moreover, $\{u(t, x_n)\}_n^{\infty}$ is uniformly convergent for $t \in \mathbb{R}$ if, in addition, the following holds:

(A1') $\sup \{ \|G(t)\| : t \in \mathbb{R} \} < \infty$.

Finally, if $x = \lim x_n$ then $x$ is the unique vector in $X$ satisfying $\mu(X) = \inf \{ \mu(y) : y \in X \}$.

The next to last assertion was proved by Zaidman [1] for the case when $X$ is a Hilbert space. The above theorem extends [1] in two ways: (i) The Hilbert space is replaced by a more general Banach space, and (ii) a weaker conclusion is established under weaker but more easily verifiable hypotheses. The original problem arose in Zaidman's work on almost periodicity (cf. [1] for references); it is hoped that the above theorem is applicable to that theory.

To prove the theorem, let $\{x_n\}_1^{\infty}$ be a minimizing sequence. We may suppose that $\mu(x_n) < \infty$ for each $n$. We claim that $\{x_n\}_1^{\infty}$ is a Cauchy sequence in $X$. Assume not. Then there is an $\varepsilon_0 > 0$ and subsequences $\{n_j\}_1^{\infty}$, $\{m_j\}_1^{\infty}$ of the positive integers such that

$$\|x_n - x_{m_j}\| > \varepsilon_0 \quad \text{for } j = 1, 2, \ldots.$$