A Statistical Generalization of the Quantum Mechanics (I).

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Summary. — It is shown that the ordinary Schrödinger equation for a dynamical system $\Sigma$ may be replaced by a more general equation, which has the form of the Schrödinger equation of a quantized Bose field whose quanta are the systems $\Sigma$. The linear wave functionals of the quantized field describe pure states of the system $\Sigma$ and the non linear wave functionals describe, in general, mixtures of states of $\Sigma$. The representation in which the emission operators of the quantized $\Sigma$ field are diagonal plays a central role in the present formalism. It is shown that the eigenfunctionals of the absorption operators of the quantized $\Sigma$ field can be used to obtain a new description of the states of a system $\Sigma$, the expectation values of the field quantities in suitably chosen eigenstates of the emission operators coinciding with the expectation values of the corresponding quantities in the pure states of $\Sigma$, but the fluctuations being larger in the former case. The eigenfunctionals of the absorption operators have the remarkable property of being matrix elements of the unity operator of the field formalism, and seem to be of a more fundamental nature than the linear functionals which correspond to the ordinary description of the pure states by means of the wave functions of $\Sigma$.

1. - Introduction.

In the ordinary form of the quantum mechanics, the states of a system are described by wave functions $\Psi(t; x)$ depending on the time $t$ and variables $x$. The time evolution of the wave function is determined by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(t; x) = H\Psi(t; x),$$

(1)
$H$ being the hamiltonian operator of the system $\Sigma$. For the sake of simplicity we shall treat only the case of continuous variables $x$, so that the $x$ will be coordinates of a point in a continuous space $\Omega$. The case, in which some or all of the $x$ are discrete variables, can be treated in a similar way, with minor modifications.

In general, we shall assume that the wave functions $\Psi(x)$ are normalizable and normalized as usual: $\int_\Omega |\Psi|^2 \, dx = 1$. These normalizable wave functions may be considered as vectors of a Hilbert space, the $\psi$-space. The complex conjugate functions $\Psi^*(x)$ are vectors of another Hilbert space, the $\psi^*$-space. We have shown (1) that it is possible to describe the motion of the system $\Sigma$ by means of the vectors of the Hilbert space dual to the $\psi^*$-space, the $\chi$-space. The vectors of the $\chi$-space correspond to the continuous linear functionals $\chi[\psi^*]$ defined in the $\psi^*$-space:

$$\chi[\psi^*] = \int_\Omega \Psi(x) \psi^*(x) \, dx.$$  

It follows from (1) and (2) that the linear functionals $\chi[\psi^*]$ built with the solutions of (1) satisfy the equation:

$$ih \frac{d}{dt} \chi[t; \psi^*] = \mathcal{H} \chi[t; \psi^*] = \int_\Omega dx \psi^*(x) \frac{\delta}{\delta \psi^*(x)} \chi[t; \psi^*],$$

which was already given in reference (1). Equation (3) may be considered as the Schrödinger equation of the dual $\chi$-space. Equation (3) admits as solutions non linear functionals, which are no more equivalent to wave functions $\Psi$. It is preferable to consider (3) as the basic equation, instead of (1), because the functionals $\chi[\psi^*]$ allow us to describe, in an effective non symbolic way, states of $\Sigma$ which can only be described by means of symbolic functions, such as the Dirac $\delta$-function. It is well known from the Schwarz theory of distributions that, in order to avoid the use of symbolic functions, it is necessary to use functionals.

Once it is assumed that (3) is the fundamental equation of evolution, it becomes natural to consider also the non linear functionals which are solutions. In this paper we shall consider only the solutions of (3) which can be expanded in Volterra series:

$$\chi[\psi^*] = \chi[0] + \sum_{n=1}^\infty \frac{1}{\sqrt{n!}} \int_\Omega \Psi_n(x_1, \ldots, x_n) \psi^*(x_1) \ldots \psi^*(x_n) \, dx_1 \ldots dx_n.$$  