NEGATIVE DEPENDENCE STRUCTURES
THROUGH STOCHASTIC ORDERING

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ABSTRACT

Several new multivariate negative dependence concepts such as negatively upper orthant dependent in sequence, negatively associated in sequence, right tail negatively decreasing in sequence, and upper (lower) negatively decreasing in sequence through stochastic ordering are introduced. These concepts conform with the basic idea that if a set of random variables is split into two sets, then one is «increasing» whenever the other is «decreasing». Our concepts are easily verifiable and enjoy many closure properties. Applications to probability and statistics are also considered.

1. INTRODUCCION AND SUMMARY

The concept of negatively dependent in sequence through stochastic ordering (NDS) random variables was introduced into the statistical literature by Block, Savits and Shaked (1985) (see also, Joag-Dev and Proschan, 1983 and references there).

In this paper we introduce several new multivariate concepts (see Section 2 for exact definitions). The main motivation for our definitions is to follow the intutive requierement that if a set of negatively dependent random variables is split into two subsets in some manner then one subset will tend to be «large» when the other subset is «small» and vice versa. In Section 2, we introduce several new types of negative dependence, and develop their properties. As will be seen in Sections 3 and 4, our conditions are often easily verifiable, they arise naturally in many
applications and enjoy some closure properties which enable us to derive useful inequalities for many well known distributions.

In the sequel the following two well known results, which are useful in their own right, will be used.

Let \((X, Y)\) be a pair of real random variables and \(Z\) be a real or vector valued random variable. Then

\[
\begin{align*}
\text{Cov} (X, Y) &= E \{ \text{Cov} (X, Y | Z) \} + \text{Cov} \{ E(X | Z), E(Y | Z) \}.
\end{align*}
\]

Let \(X\) be a real random variable. For every pair of increasing functions \(f, g:\)

\[
\text{Cov} [f(X), g(X)] \geq 0.
\]

For \(f\) and \(g\) discordant functions, the inequality is reversed. (Two functions are discordant if one is increasing and other is decreasing). Inequality (1.2) is well known as Tchebycheff's inequality.

Throughout this paper, we use «increasing» in place of «nondecreasing» and «decreasing» in place of «nonincreasing». Vectors in \(\mathbb{R}^n\) are denoted by \(x = (x_1, \ldots, x_n)\) and \(x \preceq y\) means \(x_i \leq y_i, i = 1, \ldots, n\). Similarly \(x^i\) denotes \((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\) and \(x \succ y\), means \(x_i > y_i, i = 1, \ldots, n\). A real function on \(\mathbb{R}^n\) will be called increasing if it is increasing in each variable when the other variables are held fixed.

2. NEGATIVE DEPENDENCE CONCEPTS

**Definition 2.1.** (Ebrahimi and Ghosh, 1981). The random variables \(X_1, \ldots, X_n\) (or the random vector \(X\) or its distribution function) are said to be negatively upper orthant dependent (NUOD) if for every \(x,\)

\[
P[X > x] \leq \prod_{i=1}^{n} P(X_i > x_i)
\]

They are said to be negatively lower orthant dependent (NLOD) if for every \(x,\)

\[
P[X \leq x] \leq \prod_{i=1}^{n} P(X_i \leq x_i)
\]

When \(n = 2\), (2.1a) and (2.1b) are equivalent, but not when \(n \geq 3\) (see, e.g., Ebrahimi and Ghosh, 1981). If both (2.1a) and (2.1b) are