ALMOST GLOBAL EXISTENCE
FOR SOME SEMILINEAR WAVE EQUATIONS

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1 Introduction

This article establishes the almost global existence of solutions of three-
dimensional quadratically semilinear wave equations with the use of only the
classical invariance of the equations under translations and spatial rotation. Using
these techniques, we can handle semilinear wave equations in Minkowski space or
semilinear Dirichlet-wave equations in the exterior of a nontrapping obstacle.

Our results and approach are related to previous work in the non-obstacle case.
In particular, in [2], almost global existence was shown for general, quadratic
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for obstacle problems is that the translation and spatial rotational vector fields essentially preserve the Dirichlet conditions. The homogeneous vector fields used in [2] do not have this property for any obstacle, and consequently it seems difficult to use them for nonlinear obstacle problems.

Let us now describe more precisely the equations that we consider. If \( \mathcal{K} \subset \mathbb{R}^3 \) is a smooth compact nontrapping obstacle, we consider semilinear systems of the form

\[
\begin{align*}
\Box u &= Q(u'), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K}, \\
u(t,\cdot)|_{\mathcal{K}} &= 0, \\
u(0,\cdot) &= f, \quad \partial_t u(0,\cdot) = g.
\end{align*}
\]

Here

\[
\Box = \partial^2_t - \Delta
\]

is the D'Alembertian, with \( \Delta = \partial^2_x + \partial^2_y + \partial^2_z \) being the standard Laplacian. Also, \( Q \) is a constant coefficient quadratic form in \( u' = (\partial_t u, \nabla_x u) \).

In the non-obstacle case, we obtain almost global existence for equations of the form

\[
\begin{align*}
\Box u &= Q(u'), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
u(0,\cdot) &= f, \quad \partial_t u(0,\cdot) = g.
\end{align*}
\]

In order to solve (1.1), we must also assume that the data satisfies the relevant compatibility conditions. Since these are well-known (see, e.g., [3]), we shall describe them briefly. To do so, let \( J_k u = \{ \partial^\alpha u : 0 \leq |\alpha| \leq k \} \) denote the collection of all spatial derivatives of \( u \) of order up to \( k \). Then if \( m \) is fixed and if \( u \) is a formal \( H^m \) solution of (1.1), we can write \( \partial^\alpha_t u(0,\cdot) = \psi_k(J_k f, J_{k-1} g) \), \( 0 \leq k \leq m \), for certain compatibility functions \( \psi_k \) which depend on the nonlinear term \( Q \) as well as \( J_k f \) and \( J_{k-1} g \). The compatibility condition for (1.1) with \( (f, g) \in H^m \times H^{m-1} \) is just the requirement that the \( \psi_k \), vanish on \( \partial \mathcal{K} \) when \( 0 \leq k \leq m - 1 \). Additionally, we say that \( (f, g) \in C^\infty \) satisfy the compatibility conditions to infinite order if this condition holds for all \( m \).

Let \( \{ \Omega \} \) denote the collection of vector fields \( x_i \partial_j - x_j \partial_i, 1 \leq i < j \leq 3 \). We can now state our main result.

**Theorem 1.1.** Let \( \mathcal{K} \) be a smooth compact nontrapping obstacle and assume that \( Q(u') \) is as above. Assume further that \( (f, g) \in C^\infty(\mathbb{R}^3 \setminus \mathcal{K}) \) satisfies the compatibility conditions to infinite order. Then there are constants \( c, \varepsilon_0 > 0 \) such that if \( \varepsilon \leq \varepsilon_0 \) and

\[
(1.3) \quad \sum_{|\alpha| + j \leq 10} \| \partial_x^\alpha \Omega^g f \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} + \sum_{|\alpha| + j \leq 9} \| \partial_x^\alpha \Omega^g g \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \leq \varepsilon,
\]