A NOTE ON THE USE OF MEDIAN RANGES

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(Receiver March 30, 1962)

In setting up quality control charts, the mean-range is often used to estimate the population standard deviation. E. B. Ferrel [1] has pointed out the possibility of using, instead of the mean-range, the median-range which is efficient enough for that purpose. In this paper Ferrel has given a table of expected values of the median-range for different values of \( n \), the sample size. These are asymptotic values which are obtained under the assumption that the number, \( N \), of ranges from which the median-range is obtained, is very large. In fact, they are the 50% points of the distributions of ranges from normal samples. These values also appear in [2]. In this note we examine an asymptotic formula for the expected value of the median-range in normal samples.

The author is thankful to Dr. A. Matthai for presenting the problem for investigation.

In general, the expected value of the median \( \bar{X} \) in a sample of size \( N \) from the population with probability density function \( f(x) \) is shown (see [3]) to be

\[
E(\bar{X}) = x_{0.5} \left[ 1 - \frac{1}{8(N+2)} \frac{f'(x_{0.5})}{f''(x_{0.5})} + O\left( \frac{1}{N^2} \right) \right],
\]

when \( N \) is odd,

and

\[
E(\bar{X}) = x_{0.5} \left[ 1 - \frac{1}{8(N+1)} \frac{f'(x_{0.5})}{f''(x_{0.5})} + O\left( \frac{1}{N^2} \right) \right],
\]

when \( N \) is even,

where \( x_{0.5} \) is the 50% point of the distribution.

From the tables of the probability density function of the range [4], \( f(x_{0.5}) \) have been computed by interpolation and \( f'(x_{0.5}) \) by numerical differentiation.

Let \( R \) be the median of \( N \) ranges in independent normal samples of size \( n \).

Then

\[
\frac{E(R)}{\sigma} = \begin{cases} 
\frac{d_n + \frac{e}{N+2} + O\left( \frac{1}{N^2} \right)}{\sigma}, & \text{when } N \text{ is odd}, \\
\frac{d_n + \frac{e}{N+1} + O\left( \frac{1}{N^2} \right)}{\sigma}, & \text{when } N \text{ is even},
\end{cases}
\]
where \( d_n \) and \( e \) are function of \( n \), values of which are shown in the following table 1.

In setting up quality control charts, the second term which is small may be neglected.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( d_n )</th>
<th>( e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.95387</td>
<td>0.29519</td>
</tr>
<tr>
<td>3</td>
<td>1.588</td>
<td>0.162</td>
</tr>
<tr>
<td>4</td>
<td>1.978</td>
<td>0.124</td>
</tr>
<tr>
<td>5</td>
<td>2.257</td>
<td>0.108</td>
</tr>
<tr>
<td>6</td>
<td>2.472</td>
<td>0.098</td>
</tr>
<tr>
<td>7</td>
<td>2.645</td>
<td>0.093</td>
</tr>
<tr>
<td>8</td>
<td>2.791</td>
<td>0.090</td>
</tr>
<tr>
<td>9</td>
<td>2.915</td>
<td>0.086</td>
</tr>
<tr>
<td>10</td>
<td>3.024</td>
<td>0.084</td>
</tr>
</tbody>
</table>

In case \( n=2 \), the distribution function of normal range is

\[
F(R) = 2\Phi\left(\frac{R}{\sqrt{2\sigma}}\right) - 1, \quad 0 < R < \infty,
\]

where \( \Phi(x) \) is the distribution function of standard normal distribution.

Further, if \( N \) is odd, i.e., \( N=2M+1 \),

\[
\frac{E(\hat{R})}{\sigma} = \frac{2\sqrt{2N}}{(M!)^2} \int_{-\infty}^{\infty} x[2\Phi(x) - 1]^M[2 - 2\Phi(x)]^N d\Phi(x).
\]

The values of (5) were computed for \( N=3 \) (2) 15 by numerical integration, the results of which are shown in table 2 with the approximate values based on (3). The variances and the efficiency of \( \hat{R} \) relative to the mean-range \( \bar{R} \) in the estimation of \( \sigma \) were also computed at the same time.