Exact solitary wave solutions of nonlinear wave equations

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Abstract The hyperbolic function method for nonlinear wave equations is presented. In support of a computer algebra system, many exact solitary wave solutions of a class of nonlinear wave equations are obtained via the method. The method is based on the fact that the solitary wave solutions are essentially of a localized nature. Writing the solitary wave solutions of a nonlinear wave equation as the polynomials of hyperbolic functions, the nonlinear wave equation can be changed into a nonlinear system of algebraic equations. The system can be solved via Wu Elimination or Gröbner base method. The exact solitary wave solutions of the nonlinear wave equation are obtained including many new exact solitary wave solutions.

Keywords: nonlinear wave equations, exact solitary wave solutions, travelling wave solutions, hyperbolic function method.

Nonlinear wave equations are applied in many fields of natural sciences. It is very important for us to obtain the exact solutions of these equations. Up to now, only a few methods for nonlinear wave equations proved successful, such as IST (inverse scattering transform) method, Hirota method, Bäcklund method and homogeneous balance method\cite{[1]-[4]}. Shang also carried out a deep research on how to solve nonlinear wave equations\cite{[5]}. In this paper, we present a hyperbolic function method which is based on the hyperbolic tangent method\cite{[6]}, changing a nonlinear wave equation into a nonlinear system of algebraic equations. Solving this system via Wu Elimination\cite{[7]} or Gröbner base method\cite{[8]}, the exact solutions of the nonlinear wave equation can be obtained. We apply the method to some nonlinear wave equations. The exact solutions of these equations are obtained, which indicates that the method is feasible.

1 Hyperbolic function method

The hyperbolic function method is based on the fact that many solitary wave solutions have the format of hyperbolic functions. In this method, we assume that the nonlinear wave equations have solitary wave solutions, and the solutions can be expressed as the combination of hyperbolic functions.

We assume that PDE is a nonlinear wave equation, and it can be used to describe the dynamic evolution process of solitary wave $u(x,t)$. The steps of hyperbolic function method can be shown as follows:

1) Solitary wave is a kind of special travelling wave. That PDE has travelling wave solutions requires that PDE has only one argument $\xi = kx - ct + l$, where $k$ (wave number), $c$ (frequency-
are constants to be determined and \( l \) is an arbitrary constant. Then \( u(x, t) = u(\xi) \). PDE can be changed into an ordinary differential equation (ODE) via the following differential transformation:

\[
\frac{\partial}{\partial t} = - c \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} = k \frac{\partial}{\partial \xi}.
\]  

2) In order to obtain exact solitary wave solutions of ODE, we introduce two elementary solitary wave functions \( f \) and \( g \) defined as

\[
f(\xi) = \frac{1}{\cosh \xi + r}, \quad g(\xi) = \frac{\sinh \xi}{\cosh \xi + r},
\]  

where \( r(\geq 0) \) is a constant to be determined. Functions \( f(\xi) \) and \( g(\xi) \) satisfy the coupled Riccati equations\(^9\)

\[
f'(\xi) = - f(\xi) g(\xi), \quad g'(\xi) = 1 - g^2(\xi) - rf(\xi)
\]  

and their first integral

\[
g^2(\xi) = 1 - 2rf(\xi) + (r^2 - 1)f^2(\xi).
\]  

3) We assume that the solutions of ODE are polynomials of \( f \) and \( g \) which have the polynomial degree of \( m \)

\[
\phi = \sum_{i=0}^{m} a_i f^i + \sum_{j=1}^{n} b_j f^{j-1} g,
\]  

where the coefficients \( a_i \) (\( i = 0, 1, 2, \cdots, m \)) and \( b_j \) (\( j = 1, 2, \cdots, m \)) are constants to be determined and satisfy \( a_m^2 + b_m^2 \neq 0 \). The polynomial degree \( m \) can be determined via balancing the highest order derivative terms and the nonlinear terms in ODE.

4) Constructing this polynomial with the degree \( m \) and substituting the polynomial into ODE, eliminating any derivative of \( (f, g) \) and any power of \( g \) higher than one with eqs. (3, 4) and setting the coefficients of the different powers of \( f \) and \( g \) to zero, we obtain a nonlinear system of algebraic equations (AES) with all parameters which are to be determined.

5) Solving the AES to obtain all parameters via Wu Elimination or Gröbner base method, we obtain the exact solitary wave solutions of PDE in support of computer algebra system maple 4 in this paper.

2 Exact solitary wave solutions of nonlinear wave equations

In this section we will apply the hyperbolic function method to some nonlinear wave equations to verify the correctness of the method. We wish that the exact solitary wave solutions can be obtained via the method.

2.1 Burgers equation

Burgers equation is one of the important nonlinear wave equations in physics and mechanics. Its standard format is listed as follows\(^{10}\):

\[
u_t + uu_x + pu_{xx} = 0.
\]  

In order to obtain the solitary wave solutions of eq. (6), using the hyperbolic function method we can determine the degree of the solitary wave solutions, and then we have \( m = 1 \). The corresponding AES is