Homotopy formulas and $\bar{\partial}$-equation on local $q$-convex domains in Stein manifolds*

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Abstract  The homotopy formulas of $(r, s)$ differential forms and the solution of $\bar{\partial}$-equation of type $(r, s)$ on local $q$-convex domains in Stein manifolds are obtained. The homotopy formulas on local $q$-convex domains have important applications in uniform estimates of $\bar{\partial}$-equation and holomorphic extension of CR-manifolds.

Keywords: Stein manifold, Koppelman-Leray-Norguet formula, local $q$-convex domain, homotopy formula, $\bar{\partial}$-equation.

As is well known, Stein manifold is an important manifold, where there exist many non-constant holomorphic functions. $\mathbb{C}^n$ is a Stein manifold. It is natural that one wants to study complex analysis on Stein manifolds[1]. In this paper by using Hermitian metric and Chern connection[2,3] we obtain the homotopy formulas and the solution of $\bar{\partial}$-equation on local $q$-convex domains in Stein manifolds. Local $q$-convex domain is an extension of piecewise smooth pseudoconvex domain, so the homotopy formula obtained in this paper has its general meaning, which has important applications in uniform estimates of $\bar{\partial}$-equation and holomorphic extension of CR-manifolds. Moreover in this paper we discuss $(r, s)$ differential forms on Stein manifold, which is different from $(0, s)$ differential forms. In this case one cannot use Euclidean metric as in the case of $\mathbb{C}$, since Euclidean metric is not an invariant under holomorphic transformation on Stein manifold. In order to overcome this difficulty, we have introduced Hermitian metric and Chern connection[2] and constructed various integral kernels with respect to $(r, s)$ differential forms under invariant metric on Stein manifolds, and thus obtained the above results.

Assume $X$ to be an $n$-dimensional Stein manifold. Here we still use the definitions and notations in references [1—5].

Let $D \subset X$ be a $C^{(1)}$ intersection. $(U_D, \rho_1, \cdots, \rho_N)$ is a frame for $D$. Let $\psi$ be a Leray map for the frame $(U_D, \rho_1, \cdots, \rho_N)$. Then we set

$$\phi_{OK}(z, \xi, \lambda) = \bar{\lambda}(\lambda_0) \frac{S(z, \xi)}{S(z, \xi)} + (1 - \bar{\lambda}(\lambda_0)) \phi_K(z, \xi, \lambda)$$

for $K \in P'(N)$ and $(z, \xi, \lambda) \in D \times S_K \times \Delta_{OK}$. Note that $1 - \bar{\lambda}(\lambda_0) = 0$ for $\lambda$ in the neighborhood $\Delta_{OK} \backslash \Delta_{OK}$ of $\Delta_0$, and therefore $\phi_{OK}$ is of class $C(2)$.

Now for all $K \in P'(N)$, we define Bochner-Martinelli kernel:

$$\hat{B}(z, \xi) = (-1)^{n-1}/(2\pi)^n \phi'(z, \xi) \langle S, D S \rangle \wedge (\nabla^* S, D S)^{n-1}/|S|_g^2$$

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for all \((z, \zeta, \lambda) \in D \times S_K \times \Delta_K\), and define Koppelman-Leray-Norguet kernel as

\[
\mathcal{R}_K(z, \zeta, \lambda) = (-1)^{n-1}/(2\pi)^n \varphi^*(z, \zeta) \langle \psi_{OK}, DS \rangle \wedge (\langle \Delta^\varphi \psi_{OK}, DS \rangle)^{n-1},
\]

(3)

And for all \((z, \zeta, \lambda) \in D \times S_K \times \Delta_K\), we define

\[
\mathcal{L}_K(z, \zeta, \lambda) = (-1)^{n-1}/(2\pi)^n \varphi^*(z, \zeta) \langle \psi_{OK}, DS \rangle \wedge (\langle \Delta^\varphi \psi_{OK}, DS \rangle)^{n-1},
\]

(4)

where \(\nabla^\varphi = \bar{\partial}_{z, \zeta} + \Delta^\varphi = (\bar{\partial}_{z, \zeta} + d_{\lambda})\), \(\varphi\) is a holomorphic function, \(\nu\) is a suitable integer such that \(\bar{\partial}(z, \zeta), \mathcal{R}_K(z, \zeta, \lambda), \mathcal{L}_K(z, \zeta, \lambda)\) are of continuous forms.[1-4]

Set

\[
d\mathcal{R}_K(z, \zeta, \lambda) := Q_K(z, \zeta, \lambda),
\]

(5)

\[
\bar{\partial}_{z, \zeta} \bar{\partial}(z, \zeta) := P(z, \zeta).
\]

(6)

Assume \(f\) to be a continuous \((r, s)\)-form on \(\bar{D}\). For all \(K \in P'(N)\), we set

\[
B_{Df}(z) = \int_{\zeta \in \bar{D}} f(\zeta) \wedge \bar{\partial}(z, \zeta), \quad z \in D,
\]

(7)

\[
R_{Kf}(z) = \int_{(\zeta, \lambda) \in S_K \times \Delta_K} f(\zeta) \wedge \mathcal{R}_K(z, \zeta, \lambda), \quad z \in D,
\]

(8)

\[
L_{Kf}(z) = \int_{(\zeta, \lambda) \in S_K \times \Delta_K} f(\zeta) \wedge \mathcal{L}_K(z, \zeta, \lambda), \quad z \in D,
\]

(9)

\[
Q_{Kf}(z) = \int_{(\zeta, \lambda) \in S_K \times \Delta_K} f(\zeta) \wedge \mathcal{Q}_K(z, \zeta, \lambda), \quad z \in D,
\]

(10)

\[
P_{Df}(z) = \int_{\zeta \in \bar{D}} f(\zeta) \wedge \bar{\partial}(z, \zeta), \quad z \in D.
\]

(11)

Then, for every continuous \((r, s)\)-form \(f\) on \(\bar{D}\), \(0 \leq r, s \leq n\) such that \(\bar{\partial}f\) is also continuous on \(\bar{D}\). We have the following classical Koppelman-Leray-Norguet formula[3]:

\[
(-1)^{r+s+1} f = \bar{\partial}_{z} B_{Df} - B_{D} \bar{\partial}_{z} f + \sum_{K \in P'(N)} (L_{Kf} + \bar{\partial}_{z} R_{Kf} - R_{K} \bar{\partial}_{z} f)
\]

(12)

\[
(\bar{\partial}_{z} B_{Df} - B_{D} \bar{\partial}_{z} f + \sum_{K \in P'(N)} (L_{Kf} + \bar{\partial}_{z} R_{Kf} - R_{K} \bar{\partial}_{z} f)).
\]

(13)

1. A Leray map for local \(q\)-convex domains

Let \(D \subset \subset X\) be a domain and \(\rho\) a real \(C^2\) function on \(D\). Then we denote by \(L_{\rho}(\zeta)\) the Levi form \(\rho\) at \(\zeta \in D\), and by \(F_{\rho}(\cdot, \zeta)\) the Levi polynomial of \(\rho\) at \(\zeta \in D\), i.e.

\[
L_{\rho}(\zeta) = \sum_{j, k=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \zeta_k} i_{j_k}, \quad \zeta \in D, \quad t \in \mathbb{C}^n,
\]

\[
F_{\rho}(z, \zeta) = 2 \sum_{j=1}^{n} \frac{\partial \rho(\zeta)}{\partial \zeta_j} (\zeta_j - z_j) - \sum_{j, k=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \zeta_k} (\zeta_j - z_j)(\zeta_k - z_k), \quad \zeta \in D, \quad z \in X.
\]

Moreover,

\[
\text{Re} F_{\rho}(z, \zeta) = \rho(\zeta) - \rho(z) + L_{\rho}(\zeta)(\zeta - z) + o(\text{dist}(z, \zeta)^2).
\]

Denote by \(MO(n, q)\) the complex manifold of all \(n \times n\)-matrices which define an orthogonal