A \([k, k+1]\)-factor containing given Hamiltonian cycle*

CAI Maocheng (蔡茂诚), LI Yanjun (李彦君)
(Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, China)
and M. Kano
(Department of Computer and Information Sciences, Ibaraki University, Hitachi 316, Japan)

Received November 12, 1997

Abstract Let \(k \geq 2\) be an integer and let \(G\) be a graph of order \(n\) with minimum degree at least \(k\), \(n \geq 8k - 16\) for even \(n\) and \(n \geq 6k - 13\) for odd \(n\). If the degree sum of each pair of nonadjacent vertices of \(G\) is at least \(n\), then for any given Hamiltonian cycle \(C\), \(G\) has a \([k, k+1]\)-factor containing \(C\).

Keywords: graph, connected factor, Hamiltonian cycle.

All graphs under consideration are undirected, finite and simple. A graph, denoted by \(G = (V, E)\), consists of a non-empty set \(V(G)\) of vertices and a set \(E(G)\) of edges. Let \(xy\) denote the edge joining vertices \(x\) and \(y\). If \(X\) is a subset of \(V(G)\), we write \(G[X]\) for the subgraph of \(G\) induced by \(X\), \(E(G[X]) = E(G)\) and \(\overline{X} = V(G) - X\). Sometimes \(x\) is used for a singleton \(\{x\}\).

Given a graph \(G = (V, E)\) and \(x \in V(G)\), write \(d_G(x)\) for the degree of \(x\) in \(G\), which is the number of edges of \(G\) incident to \(x\). For integers \(a\) and \(b\), \(b \geq a \geq 0\), an \([a, b]\)-factor of \(G\) is defined as a spanning subgraph \(F\) of \(G\) such that

\[ a \leq d_F(v) \leq b \quad \text{for all} \quad v \in V(G), \]

and an \([a, a]\)-factor is abbreviated to an \(a\)-factor. A subset \(M\) of \(E(G)\) is called a matching if no two edges in \(M\) are adjacent in \(G\). Other notations and definitions not defined here can be found in ref. [1].

We first mention some known results on \(k\)-factors or connected \([a, b]\)-factors.

**Theorem A**\(^{[2]}\). Let \(k\) be a positive integer, and let \(G\) be a graph of order \(n\) with \(n \geq 4k - 5\), \(kn\) even, and minimum degree at least \(k\). Then \(G\) has a \(k\)-factor if the degree sum of each pair of nonadjacent vertices is at least \(n\).

**Theorem B**\(^{[3]}\). Let \(k \geq 3\) be an integer and let \(G = (V, E)\) be a connected graph of order \(n\) with \(n \geq 4k - 3\), \(kn\) even, minimum degree at least \(k\). If for each pair of nonadjacent vertices \(u\) and \(v\) of \(V(G)\)

\[
\max\{d_G(u), d_G(v)\} \geq \frac{n}{2},
\]

\(G\) has a \(k\)-factor.

**Theorem C**\(^{[3]}\). Let \(k\) be a positive integer and let \(G\) be a graph of order \(n\) such that \(n \geq 4k - 5\).

* Project supported partially by an exchange program between the Chinese Academy of Sciences and the Japan Society for Promotion of Sciences and by the National Natural Science Foundation of China (Grant No. 19136012).
kn even, and minimum degree at least \( k \). If the degree sum of each pair of nonadjacent vertices of \( G \) is at least \( n \), then \( G \) has both a Hamiltonian cycle \( C \) and a \( k \)-factor \( F \). Hence \( G \) has a connected \([k, k + 2]\)-factor \( C + F \).

**Theorem D**\(^{[4]}\). Let \( k \geq 2 \) be an integer and \( G \) be a connected graph of order \( n \). If \( G \) has a \( k \)-factor \( F \) and, moreover, among any three independent vertices of \( G \) there are (at least) two with degree sum at least \( n - k \), then \( G \) has a matching \( M \) such that \( M \) and \( F \) are edge-disjoint and \( M + F \) is a connected \([k, k + 1]\)-factor of \( G \).

**Theorem E**\(^{[5]}\). Let \( k \geq 3 \) be an odd integer, and \( G \) be a connected graph of odd order \( n \) with \( n \geq 4k - 3 \), and minimum degree at least \( k \). If for each pair of nonadjacent vertices \( u \) and \( v \) of \( G \),

\[
\max\{d_G(u), d_G(v)\} \geq \frac{n}{2},
\]

\( G \) has an almost \( k^+ \)-factor \( F^+ \) and a matching \( M \) such that \( F^- \) and \( M \) are edge-disjoint and \( F^- + M \) is a connected \([k, k + 1]\)-factor of \( G \) (an almost \( k^+ \)-factor is a factor whose every vertex has degree \( k \) except at most one with degree \( k \pm 1 \)).

**Theorem F**\(^{[6]}\). Let \( k \geq 2 \) be an integer and let \( G \) be a graph of order \( n \) such that \( n \geq 8k - 4 \), \( kn \) is even and minimum degree at least \( n/2 \). Then \( G \) has a \( k \)-factor containing a Hamiltonian cycle.

The purpose of this paper is to extend "connected \([k, k + 1]\)-factor" in some of the above theorems to "\([k, k + 1]\)-factor containing a given Hamiltonian cycle", which is obviously a 2-connected \([k, k + 1]\)-factor under somewhat stronger conditions. Our main result is the following.

**Theorem 1.** Let \( k \geq 2 \) be an integer and let \( G \) be a graph of order \( n \geq 3 \) with minimum degree at least \( k \), \( n \geq 8k - 16 \) for even \( n \) and \( n \geq 6k - 13 \) for odd \( n \). If for each pair of nonadjacent vertices \( u \) and \( v \) of \( G \),

\[
d_G(u) + d_G(v) \geq n,
\]

then for any given Hamiltonian cycle \( C \), \( G \) has a \([k, k + 1]\)-factor containing \( C \).

**Remark 1.** The conditions \( n \geq 8k - 16 \) for even \( n \) and \( n \geq 6k - 13 \) for odd \( n \) are best possible. To see this, for even \( n \) such that \( 2k \leq n < 8k - 16 \), write \( m = (n/2) + 2 \); for odd \( n \) such that \( 2k - 1 \leq n < 6k - 13 \), write \( m = (n + 3)/2 \). Let \( C' = v_1 v_2 \cdots v_m \) be a cycle and let \( P = v_{m+1} v_{m+2} \cdots v_n \) be a path. Set \( G = C' \cup P \), where \( \cup \) denotes join union. Then it is easy to check that \( G \) has no \([k, k + 1]\)-factor containing Hamiltonian cycle \( C = v_1 v_2 \cdots v_n \) even if the minimum degree is at least \( n/2 \).

**Remark 2.** For a graph \( G \) of order \( n \), the condition that the minimum degree \( \geq n/2 \) cannot guarantee the existence of a \( k \)-factor containing a given Hamiltonian cycle in \( G \). For instance, suppose \( n \geq 5 \) and \( k \geq 3 \). Write

\[
m = \begin{cases} 
\frac{n}{2} + 2 & \text{for even } n, \\
\frac{n + 3}{2} & \text{for odd } n.
\end{cases}
\]

Let \( C' = v_1 v_2 \cdots v_m \) be a cycle and let \( P = v_{m+1} v_{m+2} \cdots v_n \) be a path. Set \( G = C' \cup P \). Then the minimum degree \( \geq n/2 \) and \( G \) has no \( k \)-factor containing Hamiltonian cycle \( C = v_1 v_2 \cdots v_n \).

**Proof of Theorem 1.** We may suppose \( k \geq 3 \) as \( G \) contains \( C \) for \( k = 2 \). Write