Eigenvalue estimate on a compact Riemann manifold

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Abstract Let $M$ be a compact Riemann manifold with the Ricci curvature $\geq -R (R = \text{const.} > 0)$. Denote by $d$ the diameter of $M$. Then the first eigenvalue $\lambda_1$ of $M$ satisfies $\lambda_1 \geq \frac{\pi^2}{d^2} - 0.52R$. Moreover if $R \leq \frac{5\pi^2}{3d^2}$, then $\lambda_1 \geq \frac{\pi^2}{d^2} - \frac{R}{2}$.

Keywords: Riemann manifold, eigenvalue, Ricci curvature.

For a compact Riemann manifold, there are many estimates about its first eigenvalue $\lambda_1$. The main results[1–4] concerned were obtained by the method of the gradient estimate of eigenfunctions introduced by Li and Yau in ref. [3]. Here we quote some important results as follows.

Theorem A[1]. Let $M$ be a compact Riemann manifold with non-negative Ricci curvature and $d$ the diameter of $M$. Then $\lambda_1 \geq \frac{\pi^2}{d^2}$.

Theorem B[2]. Let $M$ be an $m$-dimensional compact Riemann manifold with the Ricci curvature $\geq - R (R = \text{const.} > 0)$, and $d$ the diameter of $M$. Then $\lambda_1 \geq \frac{\pi^2}{d^2} \exp \left( - C_m \sqrt{Rd^2} \right)$, where $C_m = \max(\sqrt{m} - 1, \sqrt{2})$.

Theorem C[2]. Under the hypothesis of Theorem B, $\lambda_1 \geq \frac{\pi^2}{d^2} - R$.

Yang pointed out that there should be the following estimates[2]:

$$\lambda_1 \geq \frac{\pi^2}{d^2} - \frac{1}{2} R,$$

(1)

$$\lambda_1 \geq \frac{\pi^2}{d^2} \exp \left( - \frac{1}{2} C_m \sqrt{Rd^2} \right).$$

(2)

(2) was proved in ref. [4]. Some authors have been trying to improve the estimates in refs. [1,2] recent years, but their results do not seem successful. For details, see refs. [1–5]. On the basis of refs. [1,2], we shall prove the following estimates.

Theorem 1. Let $M$ be a compact Riemann manifold with the Ricci curvature $\geq - R$, $R = \text{const.} > 0$, and $d$ the diameter of $M$. Then $\lambda_1 \geq \frac{\pi^2}{d^2} - 0.52R$.

Theorem 2. Under the hypothesis of Theorem 1, if $R \leq \frac{5\pi^2}{3d^2}$, then $\lambda_1 \geq \frac{\pi^2}{d^2} - \frac{1}{2} R$.

Remark. From Theorem C and (1), we note that $R = \text{const.}$ should be smaller relative to $\pi^2/d^2$, since only in this case are the results of significance. So the hypothesis $R \leq \frac{5\pi^2}{3d^2}$ above
is reasonable.

1 Notations and formulas

In this paper we shall use the same notations as in refs. [1, 2]. Let \( M \) be an \( m \)-dimensional orientable compact Riemann manifold with its first eigenvalue \( \lambda_1 \), and let \( \{ e_i \} \) be a local orthonormal frame on \( M \) with the coframe \( \{ \omega^i \} \). Then there exist Riemann connection forms \( \{ \omega^i_j \} \) on \( M \) such that \( d\omega^i + \omega^i_j \wedge \omega^j = 0 \), \( d\omega^i_j + \omega^i_k \wedge \omega^k_j = \frac{1}{2} R^k_{ij} \omega^k \wedge \omega^i \), where \( 1 \leq i, j, k, l \leq m \), and \( R^k_{ij} = R^k_{jkl} \) are the Riemann curvature tensors of \( M \). Here and below, repeated indices mean summation.

Let \( f \) be a smooth function on \( M \). Its covariant derivatives \( f, f_i, f_{ij}, f_{ijk} \) are successively defined by \( df = f\omega^i \), \( df_i = f_i \omega^i \), \( df_{ij} = f_{ij} \omega^i \wedge \omega^j \), \( df_{ijk} = f_{ijk} \omega^i \wedge \omega^j \wedge \omega^k \). From these and the Ricci identity, we have

\[
\nabla f = f_i \omega^i,
\nabla f_i = f_{ij} \omega^i,
\nabla f_{ij} = f_{ijk} \omega^i \wedge \omega^j.
\]

The Laplacian of \( f \) is defined by \( \Delta f = \sum f_{ii} \).

Suppose \( f \) is an eigenfunction of \( \Delta \) corresponding to \( \lambda_1 \) on \( M \), i.e. \( \Delta f = -\lambda_1 f \). Since \( M \) is compact, without loss of generality we choose \( f \) such that \( \Delta f = -\lambda_1 f \), \( \lambda_1 = \max f = -k \) \((0 < k \leq 1)\). Set \( u = (2f + k - 1)(1 - \delta)/(k + 1) \), where \( \delta \) is a small enough positive constant. Then we have \( \Delta u = -\lambda_1(u + a) \), \( \max u = 1 - \delta \), \( \min u = -1 + \delta \), where \( a = (1 - k)(1 - \delta)/(1 + k) \), \( 0 < a < 1 \).

For each small enough \( \delta > 0 \), let \( \theta(x) = \arcsin u(x) \), \( x \in M \). Then \( \theta(x) \) is a smooth function. And its gradient can be written as \( \nabla \theta = \nabla u/\sqrt{1 - u^2} \). Now consider a special function \( U(\theta) \) such that

\[
U(\theta_0) \equiv \max_{\theta(x) = \theta_0} |\nabla \theta(x)|^2, \quad \forall \theta_0 \in \left[ -\frac{\pi}{2} + \delta_1, \frac{\pi}{2} - \delta_1 \right],
\]

where \( \delta_1 = \arcsin \sqrt{\delta(2 - \delta)} \). Clearly \( U \) is continuous and \( U(\pm (\frac{\pi}{2} - \delta_1)) = 0 \). Moreover \( \forall \theta_0 \in \left[ -\frac{\pi}{2} + \delta_1, \frac{\pi}{2} - \delta_1 \right], \exists x_0 \in M \) s. t. \( \theta(x_0) = \theta_0 \), \( |\nabla \theta|^2(x_0) = U(\theta_0) \).

Because we shall let \( \delta \to 0 \) (hence \( \delta_1 \to 0 \)), from now on we assume that (unless otherwise explained) \( U \) is defined on \( (-\pi/2, \pi/2) \).

2 Some lemmas

Given a compact Riemann manifold \( M \), its Ricci curvature \( \geq -R \), \( R = \text{const.} > 0 \). Let \( \lambda_1 \) \((\lambda_1 > 0)\) be the first eigenvalue of \( M \), and let \( R_1 = R/\lambda_1 \). For the function \( U(\theta) \) defined above, we have the following.

**Lemma 1.** Let \( y(\theta) \) be a \( C^2 \)-function on \( (-\pi/2, \pi/2) \) such that \( U(\theta) \leq \lambda_1 y(\theta) \), and \( U(\theta_0) = \lambda_1 y(\theta_0) > 0 \), for some \( \theta_0 \in (-\pi/2, \pi/2) \). Then at \( \theta = \theta_0 \) there holds

\[
y \leq 1 + a \sin \theta + R_1 \cos^2 \theta - y' \cos \theta \sin \theta
\]

\[
+ \frac{1}{2} y'' \cos^2 \theta - \frac{y'}{4y} [(y - 1) \cos^2 \theta]' + 2a \cos \theta. \tag{4}
\]

**Proof.** Set \( z(\theta) = \lambda_1 y(\theta) \), \( w(x) = |\nabla \theta(x)|^2 - 2a \theta \) \( \cos \theta \). By (3), \( \exists x_0 \in M \), s. t. \( \theta(x_0) = \theta_0 \), \( |\nabla \theta(x_0)|^2 = U(\theta_0) \). So the \( C^2 \)-function \( w(x) \) attains its maximum at \( x_0 \). Applying the maximum principle, we have at \( x = x_0 \), \( w(x) = 0 \), \( \nabla w = 0 \) and \( \Delta w \leq 0 \). Therefore, at \( \theta =