Operators with connected spectrum + compact operators =
strongly irreducible operators

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Abstract Herrera’s conjecture that each operator with connected spectrum acting on complex, separable Hilbert space can be written as the sum of a strongly irreducible operator and a compact operator is proved.

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Let \( L(H) \) denote the algebra of all bounded linear operators acting on a complex, separable, infinite-dimensional Hilbert space \( H \). An operator \( T \in L(H) \) is said to be strongly irreducible if it does not commute with any nontrivial idempotents\(^1\). In the above definition, if orthogonal projection replaces idempotent, then \( T \) is said to be irreducible. It is well known that reducibility is unitary invariant, but strongly irreducibility is similarly invariant.

Halmos\(^3\) proved that each \( T \) in \( L(H) \) can be written as the sum of a irreducible operator and a compact operator with small norm. If \( \sigma(T) \), the spectrum of operator \( T \), is the union of two disjoints (an infinite set and a compact set), then for arbitrary compact operator \( K \), \( T + K \) is still strongly reducible. Herrero and Jiang showed that each \( T \) in \( L(H) \) with connected spectrum can be expressed as the sum of a strongly irreducible operator and an operator with small norm\(^4\). In 1988, Herrero raised the following conjecture: Each operator with connected spectrum can be written as the sum of a strongly irreducible operator and a compact operator.

In ref. [5], Jiang et al. proved affirmatively Herrero’s conjecture for all essentially normal operators with connected spectrum. Jiang et al. proved that Herrero’s conjecture holds for all biquasitriangular operators with connected spectrum\(^1\). In ref. [6], Jiang and Guo showed that Herrero’s conjecture is true for all hoponormal operators with connected spectrum. Recently, Ji\(^2\) proved that each quasitriangular operator with connected spectrum can be written as the sum of a strongly irreducible operator and a compact operator with small norm.

In this paper, we prove Herrero’s conjecture and give the following results.

Theorem 1. Let \( T \in L(H) \), \( \sigma_0(T) \) denote the normal point spectrum of \( T \). If \( \sigma(T) \setminus \sigma_0(T) \) is connected, then there exists a compact \( K \) such that \( T + K \) is a strongly irreducible operator.


Theorem 2. Given a natural number \( n \) and a bounded connected open set of \( C \), denoted by \( \Omega \), \( B_n(\Omega) \) denotes the set of the Cowen-Douglas operators with index \( n \) in \( \Omega \). Then \( \forall T \in B_n(\Omega) \) and \( \varepsilon > 0 \), and there exists a compact \( K \) with \( \| K \| < \varepsilon \) such that \( T + K \in B_n(\Omega) \cap (SI) \), where \( (SI) \) stands for the set of all strongly irreducible operators in \( L(H) \).

1 Lemmas

Let \( \Omega \) be a non-empty bounded open subset of \( C \) such that \( (\Omega)^0 = \Omega \), where \( (\Omega)^0 \) denotes the interior of \( \Omega \). Set \( I' = \{ a \} \). Then we have

Theorem A[7].

(i) There exists a normal operator \( M(\Gamma) = \begin{bmatrix} M_+ (\Gamma) & Z \\ 0 & M_- (\Gamma) \end{bmatrix} \).

(ii) \( \sigma(M(\Gamma)) = \sigma_e(M(\Gamma)) = \sigma_e(M_+ (\Gamma)) = \sigma_e(M_- (\Gamma)) = \Gamma \),

\[ \sigma(M_+ (\Gamma)) = \sigma(M_- (\Gamma)) = \Omega, \quad \text{ind}(M_+ (\Gamma) - \lambda) = \text{ind}(M_- (\Gamma) - \lambda)^* = -1; \lambda \in \Omega, \]

\[ \text{dimker}(M_+ (\Gamma) - \lambda) = \text{dimker}(M_- (\Gamma) - \lambda)^* = 0; \lambda \in \Omega, \]

(iii) \( Z \) is a compact operator.

\( \sigma_e(\cdot) \) denotes the essentially spectrum of operator and \( \text{ind}(\cdot - \lambda) \) the Fredholm index of operator in \( \lambda \).

Theorem A shows that \( M_+ (\Gamma) \) and \( M_- (\Gamma) \) are essentially normal. Furthermore, if \( \Omega \) is connected, then \( M_+ (\Gamma) \in B_1(\Omega^*) \); \( M_- (\Gamma) \in B_1(\Omega) \), where \( \Omega^* = \{ \lambda; \lambda \in \Omega \} \).

Theorem B[1]. Given \( A, B \in L(H) \), the Rosenblum operator \( \tau_{AB} \in L(H) \) is defined by \( \tau_{AB}(X) = AX - XB \). Then the following are equivalent:

1. \( \sigma_r(A) \cap \sigma_l(B) = \emptyset \);
2. \( \tau_{AB} \) is surjective;
3. \( \text{Ran} \tau_{AB} \) contains the all-compact operator of \( L(H) \).

Lemma 1. Let \( T = \begin{bmatrix} T_1 & T_{12} \\ 0 & T_2 \end{bmatrix} \) satisfy the following conditions:

1. \( T_2 \in B_n(\Omega) \cap (SI) \).
2. \( \sigma_0(T_1) = \sigma(T_1) \cap \Omega = \{ \lambda_k \mid k \geq 1 \} \) such that \( \text{null}(T_1 - \lambda_k) = \text{dim}H_1(\lambda_k, T_{11}) = n \) and \( \forall k(1, 2, \ldots) = H_1 \) where \( H_1(\lambda_k, T_{11}) \) denotes Riesz decomposition space of \( T_0 \) on \( \lambda_k \).
3. \( B_k = P_{\ker(T_1 - \lambda_k)} : T_{12} \mid_{\ker(T_2 - \lambda_k)} \) is injective, where \( P_{\ker(T_1 - \lambda_k)} \) is the orthogonal projection onto \( \ker(T_1 - \lambda_k)^* \).

Then \( T \in B_n(\Omega) \cap (SI) \).

Proof. We claim that \( \ker(T - \lambda_k) = \ker(T_1 - \lambda_k) \). If there are \( x \in H_1 \), \( y \in H_2 \) such that \( \begin{bmatrix} (T_1 - \lambda_k) & T_{12} \\ 0 & (T_2 - \lambda_k) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \), then \( (T_1 - \lambda_k)x + T_{12}y = 0, \ (T_2 - \lambda_k)y = 0. \)

Since \( P_{\ker(T_1 - \lambda_k)} : (T_1 - \lambda_k) = 0 \), we know \( P_{\ker(T_1 - \lambda_k)} : T_{12} \gamma = 0 \). Since \( P_{\ker(T_1 - \lambda_k)} \) we know that \( T_{12} \mid_{\ker(T_1 \lambda_k)} \) is injective and \( y \in \ker(T_{12} - \lambda_k) \), \( y = 0. \) This shows \( \ker(T - \lambda_k) = \ker(T_1 - \lambda_k) \).

1) see footnote 2) on page 925.