Some relations between entropy and approximation numbers*

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Abstract A general result is obtained which relates the entropy numbers of compact maps on Hilbert space to its approximation numbers. Compared with previous works in this area, it is particularly convenient for dealing with the cases where the approximation numbers decay rapidly. A nice estimation between entropy and approximation numbers for non-compact maps is given.

Keywords: compact and noncompact linear maps, entropy numbers, approximation numbers.

Let $T$ be a compact linear map from a Banach space $X$ to another Banach space $Y$, and let $e_n(T)$ and $a_n(T)$ be its $n$th entropy number and its $n$th approximation number, respectively. In recent years a good deal of attention has been paid to the relations between the $e_n(T)$ and the $a_n(T)$. For example, Carl[1] has shown that if either $X$ or $Y$ is a Hilbert space and $a > 0$, then $a_n(T) = n^{-a}$ if and only if $e_n(T) = n^{-a}$ (see also ref. [2]) in the case where both $X$ and $Y$ are Hilbert spaces. Moreover, Triebel[3] has established a general result, in the context of arbitrary Banach (or quasi-Banach) spaces, a special case of which shows that if $a_n(T) = O(n^{-a})$ (or $O((\log n)^{-a})$), then $e_n(T) = O(n^{-a})$ (or $O((\log n)^{-a})$). His general result holds only when the approximation numbers do not decrease very rapidly (exponential decay is not allowed). In the present paper we provide some information about the case of repaid decay, in the context of a Hilbert space.

We discuss the case in which $X = Y$ and $X$ is a Hilbert space, and obtain a general theorem which relates the entropy numbers of $T$ to its approximation numbers. A feature of the proof is its use of estimates for the entropy numbers of finite-dimensional approximations of $T$. This general result is particularly convenient for dealing with cases in which the approximation numbers decay rapidly. Thus for example, if $a_n(T) = O(\exp(-\lambda n^\gamma))$ for some positive $\lambda$ and $\gamma$, then $\log e_n(T) = O(\exp(-n^{\gamma/(1+\gamma)}))$. In particular, if $a_n(T) = O(\exp(-\lambda n))$, then $\log e_n(T) = O(\exp(-n^{1/2}))$. Carl and Stephan[4] gave an example of a diagonal map $D: l_p \rightarrow l_p (1 < p < \infty)$ for which $a_n(D) = 2^{-n}$ and $e_n(D) = O(2^{-2\sqrt{2n}})$; they also showed that if $e_n(T) = O(\exp(-\lambda n))$ for some $\lambda > 0$, then $T$ is of finite rank. This paper provides various illustrations of these theorems and presents a discussion of non-compact maps.

1 Preliminaries

Throughout the paper $H$ stands for a real Hilbert space with norm $\| \cdot \|$ and inner product $(\cdot, \cdot)$; the results to be obtained will also hold, after obvious modifications, for complex spaces.

The open ball in $H$ with centre $x$ and radius $r$ is denoted by $B(x, r)$, and $\{ x \in H : \| x \| \leq 1 \}$

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by $B_H$. Let $T \in \mathcal{B}(H)$, where $\mathcal{B}(H)$ stands for the space of all bounded linear maps from $H$ to itself. Given any $m \in \mathbb{N}$, the $m$th entropy number of $T$, $e_m(T)$, is defined by

$$e_m(T) = \inf\{\varepsilon > 0 : T(B_H) \text{ can be covered by } 2^{m-1} \text{ balls in } H \text{ of radius } \varepsilon\};$$

and the $m$th approximation number of $T$, $a_m(T)$, is given by

$$a_m(T) = \inf \| T - L \|,$$

where the infimum is taken over all those bounded linear maps $L$ from $H$ to $H$ such that $\dim L(H) < m$. Following chap. V of ref. [5] for each $I \in \mathcal{M}$ we shall write

$$\omega(I) = \prod_{j=1}^{l} a_j(T). \quad (1.1)$$

The Lyapunov exponents $\mu(I)$ of $T$ are given by

$$\mu(I) = \log_2 a(I), \quad \text{if } a(I) \neq 0. \quad (1.2)$$

Now let $T \in \mathcal{K}(H)$, where $\mathcal{K}(H)$ stands for the space of all compact linear maps from $H$ to $H$. To estimate the entropy numbers of $T$ we define

$$K(\varepsilon, T) : = \text{least number of closed balls needed to cover } T(B_H) \text{ in } H \text{ of radius } \varepsilon > 0. \quad (1.3)$$

This is related to the metric entropy of $T(B_H)$. For general information about $e_n(T)$, $a_n(T)$ and $K(\varepsilon, T)$, refer to reference [4] or [6].

We recall that[6] given any $u \in H$ and any $T \in \mathcal{K}(H)$,

$$Tu = \sum_{m} a_m(T)(u, \phi_m) \psi_m, \quad (1.4)$$

where the $\phi_m$ are normalized eigenvectors of $|T|$ (the non-negative square root of $T^* T$) corresponding to the eigenvalue $a_m(T) \neq 0$ of $|T|$ and $\psi_m = T\phi_m/a_m(T)$ if $a_m(T) \neq 0$; $\phi_m = \psi_m = 0$ otherwise. For each $n \in \mathbb{N}$ we let $H_n = \text{span}\{\psi_1, \ldots, \psi_n\}$, let $P_n : H \to H_n$ be the orthogonal projection of $H$ onto $H_n$, put $T_n = T \circ P_n$, $T_n^+ = T(I - P_n)$ where $I$ is the identity map $H \to H$, and observe that $\|T_n\| = a_n+1(T)$.

2 Compact maps

Suppose that $T \in \mathcal{K}(H)$. We begin with an inequality relating the quantities $K$ and $\omega(T)$ defined in (1.1) and (1.3) (see ref. [5] or [7]). We give a short proof for convenience.

**Lemma 1.** Let $n \in \mathbb{N}$ and suppose that there is a number $\beta > 0$ such that $a_{n+1}(T) \leq \beta \leq \|T\|$. Then

$$\beta^l K(\beta \sqrt{n+1}, T) \leq 2^l \omega(T), \quad (2.1)$$

where

$$l = \max\{j \in \mathbb{N} : j \leq n, \beta \leq a_j(T)\}. \quad (2.2)$$

**Proof.** Since $T_n(B_H)$ can be identified with an ellipsoid in $\mathbb{R}^n$ with semi-axes $a_1(T), \ldots, a_n(T)$, we see that

$$T(B_H) \subset Q \times B, \quad (2.3)$$

where $Q$ is a closed box in $\mathbb{R}^n$ centred at $O$ and with sides of length $2a_1(T), \ldots, 2a_n(T)$, and $B$ is the closed ball in $H_n^\perp$ with centre $O$ and radius $a_{n+1}(T)$. Plainly $Q$ can be covered by

$$N := \prod_{j=1}^{l} \left( \left[ \frac{a_j(T)}{\beta} \right] + 1 \right)$$

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