Number of zero point for a polynomial in real field

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Keywords: polynomial, real zero point, coefficients.

It is well known that for any real coefficient polynomial \( f(x) \) of degree \( n \), how to determine its number of real zero points is a very important problem in theory of polynomial, as well as in applications in mathematics, mechanics, physics and other subjects. The way to solve this problem by using Euclidean mutual division as pointed out by Sturm⁶ is not simple and convenient for us to use the theoretic analyses for the above problem because the relation between
the number of zero points of \( f(x) \) and its coefficients has not been given directly. By choosing the \( n \) minor determinants from the eliminant of \( f'(x) \) and \( f(x) \), and considering their signs, the formula about the number of zero point of \( f(x) \) is given in this note, which is similar to the necessary and sufficient conditions about all the zero points with the negative part for a polynomial given successfully by Hurwitz\(^2\), according to the signs of some determinants.

Let
\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0, \tag{1}
g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0, \tag{2}
\]
where \( a_n b_m \neq 0, \ n \geq m \). Denote
\[
D_0(g, f) = b_m^{n-m}. \tag{3}
\]
\[
\forall k \in \{1, 2, \ldots, m\}, \text{ from the eliminant of } g \text{ and } f,
\]
\[
D(g, f) = \begin{vmatrix}
 b_m & b_{m-1} & \cdots & b_0 \\
 b_m & b_{m-1} & \cdots & b_0 \\
 \vdots & \vdots & \ddots & \vdots \\
 a_n & a_{n-1} & \cdots & a_0 \\
 a_n & a_{n-1} & \cdots & a_0 \\
 \vdots & \vdots & \ddots & \vdots \\
 a_n & a_{n-1} & \cdots & a_0
\end{vmatrix}. \tag{4}
\]
A minor determinant of order \( n - m + 2k \) is chosen, which is denoted by \( D_k(g, f) \), composed of the first \( n - m + 2k \) columns, the first \( n - m + 2k \) rows and \((n+1)\)th to \((n+k)\)th rows of \( D(g, f) \), namely
\[
D_k(g, f) = \begin{vmatrix}
 b_m & \cdots & b_{m-k+1} & \cdots & b_{2m-1-n-k} & \cdots & b_2m-1-n-2k \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 b_m & \cdots & b_{m-n} & \cdots & b_{2m-n-k} & \cdots & \vdots \\
 \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
 a_n & \cdots & a_{n-k+1} & \cdots & a_{m-k+1} & \cdots & a_{m-2k+1} \\
 \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
 a_n & \cdots & a_m & \cdots & a_{m-k} \\
\end{vmatrix}, \tag{5}
\]
where \( a_{-j} = b_{-j} = 0, \ j = 1, 2, \ldots \). Obviously
\[
D_m(g, f) = D(g, f). \tag{6}
\]

**Theorem 1.** Let \( f(x) \) be a real coefficient polynomial, \( \deg f = n \). Denote
\[
\Delta_k = D_k(f', f), \ k = 0, 1, \ldots, n-1. \tag{7}
\]
If in the ordered array \( |\Delta_0, \Delta_1, \ldots, \Delta_{n-1}| \), all the nonzero elements are \( \Delta_i, \Delta_{i_1}, \ldots, \Delta_{i_s} \), where \( 0 = i_0 < i_1 < \cdots < i_s \leq n-1 \), then \( f(x) \) has exactly \( R_n \) distinct real zero points (order is not considered), where
\[
R_n = 1 + \sum_{k=1}^{s} (-1)^{i_k} \delta_k \text{sgn}(\Delta_{i_{k-1}} \Delta_{i_k}), \tag{8}
\]
\[
\delta_k = \begin{cases} 
1, & \text{when } i_k - i_{k-1} \text{ is odd,} \\
0, & \text{when } i_k - i_{k-1} \text{ is even.}
\end{cases} \tag{9}
\]