A note on \( n \)-edge chromatic number

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All graphs appearing in this note are simple. A graph with \( p \) vertices and \( q \) edges will be called a \((p, q)\)-graph. The maximum degree of \( G \) is denoted by \( \Delta(G) \).

Let \( n \geq 2 \) be an integer. The \( n \)-edge chromatic number \( \chi'_n(G) \) of a simple graph \( G \) is the minimum cardinality of a set of colors with which one can assign the colors to the edges of \( G \) such that the edges on a path of length less than or equal to \( n \) receive different colors.

The aim of this note is to explore the bounds for \( \chi'_n(G) \) and \( \chi'_n(G) + \chi'_n(\overline{G}) \). It is quite obvious that \( \chi'_2(G) = \chi'(G) \), the edge chromatic number of \( G \). About the bounds for \( \chi'_2(G) \), Vizing showed the following conclusion.

**Lemma 1.** For any simple graph \( G \),
\[
\Delta(G) \leq \chi'_2(G) \leq \Delta(G) + 1.
\]

The bounds for \( \chi'_2(G) + \chi'_2(\overline{G}) \) are due to Vizing also.

**Lemma 2.** Let \( G \) be a graph of order \( p \). If \( p \) is even, then
\[
p - 1 \leq \chi'_2(G) + \chi'_2(\overline{G}) \leq 2(p - 1);
\]
if \( p \) is odd, then
\[
p \leq \chi'_2(G) + \chi'_2(\overline{G}) \leq 2p - 3.
\]

Let \( G \) be a \((p, q)\)-graph. If \( n \geq 3 \), then it is trivial to prove that \( \chi'_n(G) \leq q \), and \( \chi'_n(G) + \chi'_n(\overline{G}) \leq p(p - 1)/2 \).

Now we shall discuss the lower bounds for \( \chi'_n(G) (n \geq 3) \) and \( \chi'_n(G) + \chi'_n(\overline{G}) (n \geq 5) \).

A subset \( D \) of \( E(G) \) is said to be an \( n \)-edge-complete set of \( G \) if any two edges \( e_1 \) and \( e_2 \) in \( D \) are in a path of length less than or equal to \( n \) in \( G \).

**Theorem 1.** If \( D \) is an \( n \)-edge-complete set of \( G \), then \( \chi'_n(G) \geq |D| \).

The following conclusion of Lemma 2 is a direct corollary of Theorem 5.2 in reference [3].
Theorem 2. Let $G$ be a bipartite graph with bipartition $(X, Y)$. If the degree of each vertex in $X$ is greater than that of each in $Y$, then $G$ has a matching which saturates every vertex in $X$.

Theorem 3. If $G$ is a $(p, q)$-graph, then
\[ \chi'_3(G) \geq \lceil 2q/p \rceil + \lfloor 2q/p \rfloor - 1. \]  
(1)

Proof. Write $r = \lceil 2q/p \rceil$ and $s = \lfloor 2q/p \rfloor$. Let $D$ be a 3-edge-complete set of $G$, then $|D| = r + 1 > s$ by Lemma 1. Then $|L(v)| = \chi'_3(G) < s$ since $N(v)$ is a 3-edge-complete set of $G$.

Thus if $D$ is a 3-edge-complete set of $G$, then $|D| \leq \chi'_3(G) < s$ by Lemma 1. For each edge $e$, let $N(e) = \{g \in E \mid g = e \text{ or } g \text{ is adjacent to } e\}.$ Then $|N(e)| < \chi'(G) < s$ since $N(e)$ is a 3-edge-complete set of $G$.

Let $X = \{v \in V \mid d_G(v) > r\}$, then there exist no edges joining the vertices in $X$. In fact, if $e$ is an edge joining two vertices in $X$, then $|N(e)| > 2r + 1 > s$, a contradiction. Let $Y = V - X$. Denote by $(X, Y)$ the bipartite graph induced by the edges of $G$ between $X$ and $Y$. By Lemma 2, $(X, Y)$ has a matching with $|X|$ edges. Let $M$ be a matching of $(X, Y)$ with $|M| = |X|$. If a vertex $v$ is not incident with any edge in $M$, then $v \in Y$ and $d_G(v) < r$. Thus
\[ \sum_{v \in V(G)} |L(v)| \geq (d_1 + \ldots + d_p)/2. \]

Considering the extremal value of function $f(d_1, ..., d_p) = (d_1^2 + \ldots + d_p^2)/2 + q$ on condition that $d_1 + \ldots + d_p = 2q$, we have
\[ \sum_{v \in V(G)} |L(v)| \geq q(2q + p)/p. \]

Thus there exists a vertex $u$ in $G$ such that
\[ |L(u)| \geq \frac{1}{p} \sum_{v \in V(G)} |L(v)| \geq q(2q + p)/p^2 = t. \]

Hence $\chi'_n(G) \geq |L(u)| \geq t$. Therefore (2) is valid.

Denote by $d(G)$ the diameter of a graph $G$.

Lemma 3. If $d(G) \geq 4$, then $d(G) \leq 3$.

Lemma 4. If $d(G) \leq 3$, then $\chi'_n(G) = q$ when $n \geq 5$.

The proof of Lemma 4 is trivial.