K-uniformly rotund spaces and k-uniformly smooth spaces

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Abstract The conception of k-uniform smoothness (KUS) is introduced. It is the extension of the conception of uniform smoothness. It is proved that the k-uniform smoothness and Sullivan’s K-uniform rotundity (KUR) are the dual notions. X* is a KUR space if and only if X is a KUS space, X* is a KUS space if and only if X is a KUR space. If X is a KUS space, then X is a (K + 1)US space. It is also proved that the KUS space includes the Nan’s k-strongly smooth space.

Keywords: K-uniform smoothness, K-uniform convexity.

The study of conjugate relationship plays an important role in the investigation of geometry of Banach space (especially in the study of convexity and smoothness). Therefore, it would be more important for us to introduce properly and study the conception of duality of convexity C (or smoothness S) once some convexity C (or smoothness S) is introduced and widely investigated.

Since in 1977 Sullivan introduced the K-uniformly rotund space (KUR)¹¹, there have been many authors in the world studying the property of KUR space, and obtaining a lot of inspiring results, and the study of this space theory is now quite perfect. But the conception of the duality of k-uniform convexity—k-uniform smoothness has not been given out yet, and thus the study on k-uniform smoothness of Banach space has not initiated yet. Although in 1984 Istratescu introduced a kind of conception of k-uniform smoothness², his k-uniform smoothness is equivalent to the general uniform smoothness (this was proved by Nan³³).

In this note, we properly introduce the conception of k-uniform smoothness (KUS), and show that the k-uniform smoothness which we introduce is just the conception of duality of Sullivan’s k-uniform convexity, and obtain that if X is KUS space, then X is (K + 1) US space, and that X* is a KUR space if and only if X is a KUS space, X* is KUS space if and only if X is a KUR space. Therefore we have clearly known the character of dual space X* of k-uniformly rotund space. On the base of the above conception we studied the relationship between Nan’s k-strongly smooth space and k-uniformly smooth space we introduced above, and showed that KUS space includes k-strongly smooth space. While we considered the special case k = 1, we confirmed that 1-uniformly smooth space is a general uniformly smooth space. Thus, k-uniform smoothness is a natural development of uniform smoothness. This note develops and perfects the results of refs. [1—10] as well as the relative study about KUR space. As to further studies on KUS space and local k-uniformly smooth (LKUS) space we will start them in the next paper.

1 Definition of KUS spaces and main results

Definition 1. A Banach space X is said to be a k-uniformly smooth space (KUS) if for each ε > 0 there is a δ(ε) > 0 such that for x₁*, ..., xₖ₊₁* ∈ S(X*). If ∥x₁* + ... + xₖ₊₁*∥ > (k + 1)*δ, then

\[ B(x₁*, ..., xₖ₊₁*) = \sup \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x₁^*(x₁) & x₂^*(x₁) & \cdots & xₖ₊₁^*(x₁) \\ \vdots & \vdots & \ddots & \vdots \\ x₁^*(xₖ) & x₂^*(xₖ) & \cdots & xₖ₊₁^*(xₖ) \end{vmatrix} < ε. \]

Theorem 1. (a) If X* is a KUR space, then X is a KUS space. (b) If X* is a KUS space, then X is a KUR space.

Theorem 2. X* is a KUR space if and only if X is a KUS space.

Theorem 3. X* is a KUS space if and only if X is a KUR space.

Theorem 4. If X is a KUS space, then X is a k-strongly smooth space.

Corollary 1. X* is uniformly smooth if and only if X is uniformly convex⁴⁵.

Corollary 2. If X is a uniformly smooth space, then X is a strongly smooth space.

References


Theorem 5. If $X$ is a KUS space, then $X$ is a $(K+1)$US space.

Example. There exists an infinite-dimension KUS space $X$ which is not $(K-1)$US. Let $k \geq 2$ be an integer, and let $i_1 < i_2 \cdots < i_k$. For each $x = (a_1, a_2, \cdots)$ in $l_2$, define

$$\|x\|_{i_1, \ldots, i_k}^2 = \sum_{j=1}^{k} |a_{i_j}|^2 + \sum_{i=i_1}^{i_2} |a_i|^2.$$ 

Then space $X_{i_{1}, \ldots, i_{k}} = (l_2, \| \cdot \|_{i_{1}, \ldots, i_{k}})$ is KUR\(^{[10]}\) but it is not $(K-1)$UR. By Theorems 1 and 3 we know that $X_{i_{1}, \ldots, i_{k}}$ is a KUS but not $(K-1)$US.

Remark. (i) Theorem 2 shows that if $X$ is a KUS space, then $X$ is reflexive. (ii) When $k = 1$, 1UR space is a general UR space\(^{[1]}\), 1-strongly-smooth space is general strongly-smooth space\(^{[4]}\), and 1US space is a general US space (in fact, by Definition 1 we know that $X$ is 1US if and only if for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that for $x^*, y^* \in S(X^*)$. If $\|x^* + y^*\| > 2-\delta$, then $\sup \{ |x^*(x) - y^*(x)| \} < \epsilon, \forall x \in S(X)$, also if and only if $X^*$ is uniformly rotund. Hence, $X$ is 1US if and only if $X$ is uniformly smooth).

2 Proof of theorems

For the convenience of proving Theorem 1, let us recall the definition of KUR.

A Banach space $X$ is said to be a $k$-uniformly rotund space (KUR) if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that for $x_1, \cdots, x_{k+1} \in S(X)$, if $\|x_1 + \cdots + x_{k+1}\| > (k+1) \cdot \delta$, then

$$A(x_1, \cdots, x_{k+1}) = \sup \left\{ \begin{array}{ccc} 1 & 1 & \cdots & 1 \\ x_1^*(x_1) & x_1^*(x_2) & \cdots & x_1^*(x_{k+1}) \\ \vdots & \vdots & \ddots & \vdots \\ x_k^*(x_1) & x_k^*(x_2) & \cdots & x_k^*(x_{k+1}) \end{array} : x_1^*, \cdots, x_k^* \in S(X^*) \right\} < \varepsilon.$$ 

Proof of Theorem 1. (a) If for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that for $x_1^*, \cdots, x_{k+1} \in S(X^*)$, if $\|x_1^* + \cdots + x_{k+1}\| > (k+1) \cdot \delta$, then by the assumption that $X^*$ is KUR, we have $A(x_1^*, \cdots, x_{k+1}^*) < \varepsilon$. On the other hand, for any $x_1, \cdots, x_k \in S(X)$, we have $x_1, \cdots, x_k \in S(X^*)$, then $B(x_1^*, \cdots, x_k^*) \leq A(x_1^*, \cdots, x_{k+1}^*) < \varepsilon$, and therefore $X$ is a KUS space.

(b) If for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that for $x_1, \cdots, x_{k+1} \in S(X)$, if $\|x_1 + \cdots + x_k + \cdots + x_{k+1}\| > (k+1) \cdot \delta$, by the assumption that $X^*$ is a KUS, we have $B(x_1, \cdots, x_{k+1}) < \varepsilon$, hence $A(x_1, \cdots, x_{k+1}) = B(x_1, \cdots, x_{k+1}) < \varepsilon$, therefore $X$ is KUR space.

To prove Theorem 2 we will use James’ well-known conclusion as follows.

Lemma 1\(^{[9]}\) (James). Banach space $X$ is not reflexive if and only if for each $\theta, 0 < \theta < 1$, there exists $x_\theta \in S(X), x_\theta^* \in S(X^*)$ such that

$$A_j(x_\theta) = \begin{cases} \theta & \text{if } j \leq i, \\ 0 & \text{if } j > i. \end{cases}$$

Proof of Theorem 2. Necessity. It will be obtained immediately from Theorem 1.

Sufficiency. Firstly, we prove that $X$ is reflexive. Suppose that $X$ is not reflexive. Using Lemma 1, for each $\varepsilon > 0 \ (0 < \varepsilon < 1)$, we choose $0 < \theta < 1, \theta > 1 - \frac{\delta(\varepsilon)}{k+1}$ and $\theta^k \varepsilon, x_1, \cdots, x_{k+1} \in S(X), x_1^*, \cdots, x_{k+1}^* \in S(X^*)$ so that

$$A_j(x_\theta) = \begin{cases} \theta & \text{if } j \leq i, \\ 0 & \text{if } j > i. \end{cases}$$

Here $\delta(\varepsilon)$ is the function required in the definition of KUS.

Now we have

$$\|x_1^* + \cdots + x_{k+1}^*\| \geq x_1^*(x_{k+1}) + \cdots + x_{k+1}^*(x_{k+1}) = (k+1) \theta > (k+1) \cdot \delta(\varepsilon).$$

On the other hand it is easy to check.