Prescribing symmetric scalar curvature on $S^2$

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Abstract The problem of prescribing scalar curvature in $S^2$, and the solvability of the equation $-\Delta u + 2 - Re^u = 0$ on $S^2$, is given, where $R \in C^2(S^2)$. It is known that there are some obstructions. Some new results are given by seeking a solution of the minimax type. For example, supposing that $R$ is G symmetric and constant on the set of fixed points on $S^2$ under G (where G is a subgroup of $O(3)$), it is proved that the equation is solvable if and only if $R$ is positive somewhere.

Keywords: scalar curvature, critical point, symmetric function, minimax solution, transformation group.

1 Problem and results

Given a continuous function $R$ on the standard sphere $S^2$, an interesting problem arises, i.e. the so-called Nirenberg’s problem as to whether $R$ can be the scalar curvature of some metric $\tilde{g}$ which is pointwise conformal to the standard metric $g_0$ on $S^2$. If we set $\tilde{g} = e^u g_0$, where $u$ is a function on $S^2$, the problem is then equivalent to the solvability of the following PDE:

$$-\Delta_g u + 2 - Re^u = 0 \quad \text{on} \quad S^2.$$  \hspace{1cm} (1.1)

Kazdan and Warner\(^1\) pointed out that it may be insolvable for some $R$. In the last few years, a lot of work has been done to solve this problem, especially when $R$ possesses symmetries of some kind. After the pioneer work due to Moser\(^2\) for the case of the radial symmetry, Hong\(^3\) considered the case of axisymmetry, Chang and Yang\(^4\) considered the case of reflectional symmetry w.r.t. a plane. For the case of general symmetries, Chen and Ding\(^5\) obtained some solvability results which include all the above results as special cases. In this note, we give some new solvability results of the case of general symmetries.

Let $G$ be a subgroup of the orthogonal transformation group in $\mathbb{R}^3$. Denote $S^2 = \{ x \in \mathbb{R}^3 : |x| = 1 \}$, and $f_G = \{ x \in S^2 : gx = x, \forall g \in G \}$, i.e. the set of fixed points on $S^2$ under the action of $G$. Throughout this note, suppose that $\max R > 0$, and

$$R(gx) = R(x), \quad \forall x \in S^2, \quad g \in G.$$  \hspace{1cm} (1.2)

Since $G$ here is allowed to be an arbitrary subgroup, the set $f_G$ may be $S^2$ itself (if $G$ is the unit group), or an equator, or a pair of poles, or even empty. When $G$ is the unit group, (1.2) is not a restriction at all, then on $R$ actually there is not any symmetry assumption. When $G$ is the group generated by the reflection w.r.t. $XY$-plane, for example, $f_G$ is the equator $S^2 \cap \{ |z| = 0 \}$, (1.2) means that $R$ is reflectional symmetric w.r.t. $XY$-plane. If $G$ is the group consisting of all the rotations around $Z$-axis or a discrete subgroup of it, $f_G$ only contains south and north poles in both cases, and $R$ is axisymmetric in the former case, but is, of course, not necessarily axisymmetric in the latter case. When $G = \{ \text{id.}, -\text{id.} \}$, $f_G$ is empty and $R$ is radial symmetric.

**Theorem 1.1.** Suppose that $R$ is constant on $f_G$. Then (1.1) has a solution.

**Remark 1.1.** In the case where $f_G$ is empty, undoubtedly the assumption that $R$ is constant on $f_G$ holds. So this theorem includes a result of Chen and Ding\(^5\) and the celebrated result due to Moser\(^2\) as special case. When $f_G = S^2$, the theorem yields a result known as $R \equiv \text{const}$. In the case where $f_G$ is an equator or a pair of poles the theorem gives new results.

It is worth while to mention that the argument in the proof is somewhat unusual. In short, the existence is derived from the nonexistence. In the previous work, as mentioned above, some assumptions on $R$ are imposed so that the functional has a minimum in the class of the symmetric functions, while our proof needs the use of the nonexistence of such a minimum on the contrary.

**Theorem 1.2.** Let $R \in C^2(S^2)$, and let $R_0 = \min |R(x) : x \in f_G| > 0$. If there exists a point $x_0 \in f_G$ such that $R(x_0) = R_0$ and $\Delta R(x_0) < 0$, then (1.1) is solvable.

**Remark 1.2.** Chen and Ding\(^5\) proved the solvability of (1.1) under the assumption that
max |R(x)| = \max_{x \in f_G} R(x) > 0 and there exists a point \( x_0 \in f_G \) such that \( R(x_0) = \max_{x \in f_G} R(x) \) and \( \Delta R(x_0) > 0 \). By comparing this result with Theorem 1.2, it is interesting to note that there seems to be a kind of duality. In fact it reflects some intrinsic property of the problem (see Remark 2.1 below). It should be pointed out that the assumption on the symmetry (i.e. (1.2)) is important here, since there is not such an \( R \) satisfying the conditions involved if \( G = \{ \text{id.} \} \). However, notice that Theorem 1.2 is in fact derived from a more general theorem, i.e. Theorem 2.1 below, in which the symmetry assumption (1.2) is not essential, for \( G \) is allowed to be the unit group there.

Our solution is, of course, not a minimum, but a minimax type solution. In the case where \( R \) is axisymmetric, Xu and Yang \( ^6 \) also found a solution of minimax type. But the idea is quite different. And their method is valid only in the case of axisymmetry and the nondegeneracy condition on \( R \) is required.

When \( R \) is not necessarily symmetric, there also has been much development (see refs. [7–10] and references therein).

2 A more fundamental theorem

In fact we prove a more fundamental theorem (Theorem 2.1) which asserts that a minimax value is critical. The above Theorems 1.1 and 1.2 are derived from it. Denote

\[ H_0 = \left\{ u \in H^1(S^2) : \int_{S^2} u dA = 0 \right\}, \]

Let \( \| u \| = \left( \int_{S^2} |\nabla u|^2 dA \right)^{1/2} \) be the norm of \( u \) in \( H_0 \). Denote

\[ H_* = \left\{ u \in H_0 : \int_{S^2} Re^u dA > 0 \right\}, \]

which is nonempty and open in \( H_0 \). Consider the functional \( J \) defined by

\[ J(u) = \begin{cases} \frac{1}{2} \| u \|^2 - 8\pi \log \int_{S^2} Re^u dA, & u \in H_*, \\ + \infty, & u \in H_0 \setminus H_* \end{cases} \]

It is known that, if \( u \in H_* \) is a critical point of \( J \), then \( u + c \) solves (1.1) for some constant \( c \). So it suffices to find the critical points of \( J \). Consider the class of the symmetric functions

\[ X = \left\{ u \in H_0 : u(gx) = u(x), \ \forall x \in S^2, \ g \in G \right\}. \]

Let

\[ X^\bot = \left\{ u \in H_0 : \int_{S^2} \nabla u \cdot \nabla v = 0, \ \forall v \in X \right\}, \]

the orthogonal complement of \( X \) in \( H_0 \). Making use of (1.2), the symmetry of \( R \), we easily see that automatically holds

\[ \langle dJ(u_0), v \rangle = 0, \ \forall v \in X^\bot, \]

if \( u_0 \in X \). Thus we only need to find the critical points of the restriction \( J \vert_X \), where \( X_* = H_* \cap X \).

Denote \( B_1 = \{ x \in \mathbb{R}^3 : |x|^2 < 1 \} \). Define the continuous map \( P : H^1(S^2) \to B_3 \) as

\[ P(u) = \int_{S^2} x e^{u(x)} dx / \int_{S^2} e^{u(x)} dx, \ \forall u \in H^1(S^2), \]

which is the mass center of \( u \). It is known that, for any \( u_0 \in X_* \), in the subset

\[ V(u_0) = \{ u \in X_* : P(u) = P(u_0) \}, \]

the functional \( J \) has a minimum.

Theorem 2.1. Suppose that \( R_0 = \min_{x \in f_G} |R(x)| > 0 \). If there exists \( u_0 \in X_* \) such that \( \min_{u \in V(u_0)} J(u) > -8\pi \log 4\pi R_0 \), then \( J \) has a critical point in \( X_* \) and (1.1) is solvable.

Remark 2.1. It can be easily proved that

\[ \inf_{u \in X_*} \min_{u \in V(u_0)} J(u) \leq -8\pi \log 4\pi (\max_{u \in V(u_0)} R), \]

\[ \leq -8\pi \log 4\pi (\min_{u \in V(u_0)} R) \leq \sup_{u \in X_*} \min_{u \in V(u_0)} J(u). \]

Ref. [5] actually proved that, if the first inequality is strict, then the infimum of the functional \( J \vert_X \) is