The Bergman kernels on Cartan-Hartogs domains

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Abstract The main point is the calculation of the Bergman kernel for the so-called Cartan-Hartogs domains. The Bergman kernels on four types of Cartan-Hartogs domains are given in explicit formulas. First by introducing the idea of semi-Reinhardt domain is given, of which the Cartan-Hartogs domains are a special case. Following the ideas developed in the classic monograph of Hua, the Bergman kernel for these domains is calculated. Along this way, the method of "inflation", is made use of due to Boas, Fu and Straube.

Keywords: Cartan domain, Bergman kernel function, Cartan-Hartogs domain.

The Bergman kernel function plays a very important role in several complex variables. But there are very few types of domains where one can get the Bergman kernels in explicit formulas. Hua[1] got the Bergman kernel functions for the Cartan domains of four types; Yin[2] got the Bergman kernels for two exceptional Cartan domains (Cartan domain is also called classical or symmetric domain). For the bounded homogeneous nonsymmetric domain, if its transitive group is known, one can get its Bergman kernel in explicit formulas by using Hua’s method[3, 4]. Besides these bounded homogeneous domains, one can get the Bergman kernel in explicit formulas for egg domain \( E(p_1, \cdots, p_n) \) (also called Reinhardt domain), which is defined by inequality \(|Z_1^{1/p_1} + \cdots + |Z_n|^{1/p_n} < 1\). Bergman[5] computed the Bergman kernel for \( E(1, p) \) in \( C^2 \) (although he stated that \( p \) is an integer, his computation is valid for arbitrary positive \( p \)). D’Angelo used a method to sum the orthonormal series to get the explicit formula of Bergman kernel for \( E(1, \cdots, 1, p_n) \), where \( Z \in C^i \) or \( Z \in C^n \). If \( p_1, \cdots, p_n \) are positive integers, the explicit formula of Bergman kernel for \( E(p_1, \cdots, p_n) \) was firstly computed by Zinov’ev[6]. If \( p_1, \cdots, p_n - 1 \) are positive integers and \( p_n \) is any positive real number, then \( E(p_1, \cdots, p_n) \) has Bergman kernel given by the infinite sum which is computed by Francesc and Hanges[7]. Recently, in their paper “The Bergman kernel function: explicit formulas and zeroes” (to appear in Proceedings of the AMS), H. Boas, Siqi Fu and J. Straube prove two principles (the principles of deflation and inflation), and by using the principle of folding they obtain new Bergman kernel from the old one. The principle of folding is the Bell’s transformation rule[8], which is a “useful tool to establish explicit formulas for the Bergman kernel”[9]. We cannot got the explicit formula of Bergman kernel function on any Egg domain. So we need to estimate the Bergman kernel function of Egg domains. Sheng Gong and Xuean Zheng have done that work; the Chinese version of their paper has been published, and the English version[10] of their paper published. Chieh-hsien Tiao, in his thesis of doctoral degree (Purdue Univ., 1997–1998), got the estimation of the Bergman kernel function on Reinhardt domains in general[11].

During Yin’s stay in Institut des Hautes Etudes Scientifiques (IHES) in February, 1998, he and Prof. C. Roos introduce the following four types of domains which can be called super-Cartan domains or Cartan-Hartogs domains:

\[ Y_1(N, m, n; K) := \{ W \in C^n, Z \in R_1(m, n); \| W \|^{2K} < \det(I - ZZ^T), K > 0 \}, \]

\[ Y_2(N, p; K) := \{ W \in C^n, Z \in R_2(p); \| W \|^{2K} < \det(I - ZZ^T), K > 0 \}, \]

\[ Y_3(N, q; K) := \{ W \in C^n, Z \in R_3(q); \| W \|^{2K} < \det(I - ZZ^T), K > 0 \}, \]

\[ Y_4(N, n; K) := \{ W \in C^n, Z \in R_4(n); \| W \|^{2K} < 1 - ZZ^T - [(ZZ^T)^2 - 1 ZZ^T]^{1/2}, K > 0 \}, \]

where \( R_1(m, n) \), \( R_2(p) \), \( R_3(q) \), and \( R_4(n) \) denote the Cartan domains of the first, second, third and fourth type in the sense of L. K. Hua respectively. \( Z^T \) denotes the conjugate and transport of \( Z \), “\( \det \)” denotes “determinant”, and \( N, m, n \) are positive integers.
Obviously, $Y_1(1,1,n;K) = E(1,\cdots,1,1/K)$. And if $W = 0$, then $Y_1(N,m,n;K)$, $Y_\mu(N,p;K)$, $Y_\nu(N,q;K)$ and $Y_\upsilon(N,n;K)$ become $R_1(m,n)$, $R_\mu(p)$, $R_\nu(q)$ and $R_\upsilon(n)$ respectively. Therefore, to study the super-Cartan domain is very important. If one can get the Bergman kernel in explicit formula for one domain, then this domain is a good domain for research. In this note we can get the Bergman kernel function in explicit formulas for the above four Cartan-Hartogs domains.

If one can compute the Bergman kernel functions of super-Cartan domains for $N = 1$, then by using the principle of inflation one can get the Bergman kernel functions of super-Cartan domains for $N$ in general. Therefore, we consider the case of $N = 1$ firstly. In the case of $N = 1$, we cannot use the above three principles to get the Bergman kernels in explicit formula. As the super-Cartan domains are not homogeneous, one cannot use the holomorphic automorphism group to get the Bergman kernel. The super-Cartan domains also are not the Reinhardt domain, so one cannot use the method shown in ref. [7] to compute the Bergman kernel. We use a new method to get the Bergman kernel function in explicit formula for these four Cartan-Hartogs domains. We obtain the following results. The detailed proof will appear elsewhere.

1 Group of biholomorphic automorphism

(1) The following mappings belong to the group of biholomorphic automorphism of $Y_1(1,m,n;K)$, which is denoted by Aut($Y_1$):

$$
W^* = e^{i\theta}W[I - Z_oZ_o^T]^{1/2K} \det(I - ZZ^T)^{-1/2K},
$$

$$
Z^* = A(Z - Z_o)(I - Z_o^T)D^{-1},
$$

where $A^T = (I - Z_oZ_o^T)^{-1}$, $D^T = (I - Z_o^T)Z_o^{-1}$, $Z_o \in R_1(m,n)$, $i = (-1)^{1/2}$. This mapping maps $(W,Z_o)$ onto $(W^*,0)$.

(2) The following mappings belong to the group of biholomorphic automorphism of $Y_\mu(1,p;K)$, which is denoted by Aut($Y_\mu$):

$$
W^* = e^{i\theta}W[I - Z_oZ_o^T]^{1/2K} \det(I - ZZ^T)^{-1/2K},
$$

$$
Z^* = A(Z - Z_o)(I - Z_o^T)D^{-1},
$$

where $A^T = (I - Z_oZ_o^T)^{-1}$, $D^T = (I - Z_o^T)Z_o^{-1}$, $Z_o \in R_\mu(p)$, $i = (-1)^{1/2}$. This mapping maps $(W,Z_o)$ onto $(W^*,0)$.

(3) The following mappings belong to the group of biholomorphic automorphism of $Y_\nu(1,q;K)$, which is denoted by Aut($Y_\nu$):

$$
W^* = e^{i\theta}W[I - Z_oZ_o^T]^{1/2K} \det(I - ZZ^T)^{-1/2K},
$$

$$
Z^* = A(Z - Z_o)(I - Z_o^T)D^{-1},
$$

where $A^T = (I - Z_oZ_o^T)^{-1}$, $D^T = (I - Z_o^T)Z_o^{-1}$, $Z_o \in R_\nu(q)$, $i = (-1)^{1/2}$. This mapping maps $(W,Z_o)$ onto $(W^*,0)$.

(4) The following mappings belong to the group of biholomorphic automorphism of $Y_\upsilon(1,n;K)$, which is denoted by Aut($Y_\upsilon$):

$$
W^* = e^{i\theta}W[I - X_oX_o^T]/2, (1 - ZZ^T)/(2i)), \quad Z^* = A(1,i)^{-1} \det(I - ((1 + ZZ^T)/2), (1 - ZZ^T)/(2i)) D_o^{-1},
$$

where $A' = (I - X_oX_o^T)^{-1}$, $DD' = (I - X_oX_o^T)^{-1}$, $X_o$ is the $(2,n)$ real matrix as follows:

$$
X_o = (-1/(1 - Z_oZ_o^T)), \quad Z_o \in R_\upsilon(n)\quad (i = (-1)^{1/2},$