RICCATI'S DIFFERENTIAL EQUATION IN BIRTH-DEATH PROCESSES

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This note reviews the occurrence of Riccati's equation in three birth-death type processes, and outlines their solutions.

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1. THE BIRTH-DEATH PROCESS WITH TIME-DEPENDENT PARAMETERS.

Suppose that a linear birth-death process $X(t)$ has the birth and death rates $\lambda(t)$, $\mu(t)$ respectively, where $t \geq 0$ denotes time. Then, if the p.g.f. of the process is

$$\Phi(s;0,t) = \sum_n P_{\ln}(0,t) s^n, \quad |s| \leq 1,$$

where $P_{\ln}(0,t) = \Pr\{X(t) = n | X(0) = 1\}$, this is known to satisfy the partial differential equation
\[ \frac{\partial \Phi}{\partial t} = \{ \lambda(t) s - \mu(t) \} \{ s-1 \} \frac{\partial \Phi}{\partial s} \] (1.1)

with initial condition \( \Phi(s; 0,0) = s \). The auxiliary equations are

\[ \frac{dt}{-1} = \frac{ds}{\{ \lambda(t) s - \mu(t) \} \{ s-1 \}} = \frac{d\Phi}{0} \] (1.2)

In the non-trivial case \( \lambda(t) = c \mu(t) \), where \( c \) is some positive constant, Kendall (1948) noted that

\[ \frac{ds}{dt} = -\lambda(t) s^2 + \{ \lambda(t) + \mu(t) \} s - \mu(t) \] (1.3)

was a Riccati equation for which a general solution was available. Writing \( s = 1 + w^{-1} \), Kendall effectively found

\[ \frac{dw}{dt} = \{ \lambda(t) - \mu(t) \} w + \lambda(t) \]

with the solution

\[ w e^{-\rho(t)} - \int_{(0,t)} \lambda(v) e^{-\rho(v)} dv = c_1 \]

or

\[ (s - 1)^{-1} e^{-\rho(t)} - \int_{(0,t)} \lambda(v) e^{-\rho(v)} dv = c_1 \] (1.4)

where \( \rho(t) = \int_{(0,t)} \{ \lambda(u) - \mu(u) \} du \), \( \rho(v) = \int_{(0,v)} \{ \lambda(u) - \mu(u) \} du \) and \( c_1 \) is an arbitrary constant.

The p.g.f. of the process is thus given by

\[ \Phi(s; 0, t) = f( (s - 1)^{-1} e^{-\rho(t)} - \int_{(0,t)} \lambda(v) e^{-\rho(v)} dv ) , \]

where \( f(\cdot) \) is an arbitrary function such that \( \Phi(s; 0, 0) = f( (s - 1)^{-1} ) = s \). It is readily found that

\[ \Phi(s; 0, t) = \left\{ A(t) + (1- A(t)- B(t)) s \right\} / \left\{ 1- sB(t) \right\} , \] (1.5)

where

\[ B(t) = \int_{(0,t)} \lambda(v) e^{-\rho(v)} dv \{ e^{-\rho(t)} + \int_{(0,t)} \lambda(v) e^{-\rho(v)} dv \}^{-1} , \]