A NOTE ON RIESZ AND NÖRLUND MEANS

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SUMMARY

In a previous paper [3], the author has considered the necessary and sufficient conditions in order that \((R, \lambda_n, 1) \Rightarrow (N, p_n)\) and \(|R, \lambda_n, 1| \Rightarrow |N, p_n|\). In the present note in § 2, necessary and sufficient conditions in order that \((N, p_n) \Rightarrow (R, \lambda_n, 1)\) and \(|N, p_n| \Rightarrow |R, \lambda_n, 1|\), have been discussed. Limitation theorems and the problem of ineffectiveness for ordinary and absolute summability by Nörlund means generated by a class of sequences \(\{p_n\}\), figure in § 3, and discussions on “unified theorems of consistency”, for Riesz summability form the subject matter of the last section (§ 4) of this note.

1.1. Let \(\sum a_n\) be an infinite series and let \(\{s_n\}\) be the sequence of its partial sums. Given a sequence \(\{\lambda_n\}\) such that \(\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty\), as \(n \to \infty\), the series \(\sum a_n\), or the sequence \(\{s_n\}\), is said to be summable \((R, \lambda_n, k), k \geq 0\), to zero, if

\[
A_k^\lambda(\omega) = \sum_{\lambda_{n} < \omega} (\omega - \lambda_n)^{k}a_n = o(\omega^k)(\omega \to \infty),
\]

and summable to zero by the discontinuous Riesz means \((R^*, \lambda_n, k), k \geq 0\), if \(A_k^\lambda(\lambda_n) = o(\lambda_n^k)\). Also the series \(\sum a_n\), or the sequence \(\{s_n\}\), is said to be absolutely summable \((R, \lambda_n, k)\), or summable \(|R, \lambda_n, k|\) if

\[
\frac{A_k^\lambda(\omega)}{\omega^k}
\]

is a function of bounded variation on \((\lambda_c, \infty)\), and similarly, summable \(|R^*, \lambda_n, k|\).
if the sequence
\[ r_n^k = \left\{ \frac{A_n^k(\lambda_n)}{\lambda_n^k} \right\} \]
is of bounded variation, i.e. if \( \Sigma |r_n^k - r_{n+1}^k| < \infty \).

It is now well known that, for both, the ordinary and absolute summability, the \((R^*, \lambda_n, 1)\)-mean of a series is equivalent to its \((R, \lambda_n, 1)\)-mean. In the present note we are concerned with \((R, \lambda_n, 1)\)-mean of a series and for sake of convenience, we adopt the method of discontinuous means.

For \( k = 1 \), we have,
\[ r_n^1 = r_n = \frac{1}{\lambda_n} \sum_{\nu=0}^{n} \mu_{\nu} s_{\nu}, \quad \text{where} \quad \mu_{\nu} = \lambda_{\nu} - \lambda_{\nu-1}, \quad \lambda_{-1} = 0. \]

Let \( \{p_n\} \) be a sequence of numbers, real or complex, such that
\[ p_n = p_0 + p_1 + \cdots + p_n \neq 0, \quad \text{for} \quad n \geq 0, \]
and define
\[ t_n = \frac{1}{p_n} \sum_{\nu=0}^{n} p_{n-\nu} s_{\nu}. \]

The series \( \Sigma a_n \), or the sequence \(|s_n|\), is said to be summable \((N, p_n)\) to the sum \( s \), if \( \lim_{n \to \infty} t_n = s \), and absolutely summable \((N, p_n)\) i.e. summable \(|N, p_n|\), if, in addition, \(|t_n|\) is a sequence of bounded variation.

Necessary and sufficient conditions for the regularity of the \((N, p_n)\)-method are
\[ p_n = o(|P_n|), \quad \text{as} \quad n \to \infty, \]
and
\[ \sum_{\nu=0}^{n} |p_{\nu}| = O(|P_n|), \quad \text{as} \quad n \to \infty. \]

For absolute regularity of the method, necessary and sufficient conditions are (3) and
\[ \sum_{n=0}^{\infty} \left| \frac{P_{n-k}}{P_n} - \frac{P_{n-1-k}}{P_{n-1}} \right| < K, \quad (1) \]
k = 0, 1, 2, \ldots, and \( K \) a constant independent of \( k \) (cf. [10], [8] and [13]).

(1) Throughout the paper, \( K \) stands for an absolute constant, not necessarily the same at subsequent occurrences.