VARIATION OF DIAMETRICALLY SYMMETRIC OR ELLIPTICALLY SCHLICHT CONFORMAL MAPPINGS

By

REINER KÜHN AU

Kurt Strebel zum 80. Geburtstag gewidmet

Abstract. A domain is called diametrically symmetric if it contains with each point its antipodal point on the Riemann sphere. We derive a variational formula for schlicht conformal mappings of such domains onto domains of the same type. This gives an analogue of a classical variational formula of Duren and Schiffer, which is in some sense an “elliptic analogue” of the “hyperbolic case” of Duren and Schiffer.

1 Introduction

Duren and Schiffer [D/Sch] developed a variational method (boundary variation) to solve extremal problems for classes of schlicht conformal mappings of one subdomain of the unit disk onto another. (They considered explicitly only certain special cases, typically the mappings of a ring domain onto a subdomain of the unit disk with certain normalizations; but, in fact, the considerations in [D/Sch] work much more generally.)

If one views the unit disk as the hyperbolic plane, then the mappings are those of one subdomain of the hyperbolic plane onto another subdomain. This point of view leads us to the question of an “elliptic analogy”. The classical elliptic plane (cf. F. Klein [Kl], pp. 151–153) is represented as the Riemann sphere (of diameter 1 tangent to the complex plane z in the point z = 0) with identification of antipodal points. The natural metric is the spherical metric, that is $|dz|/(1 + |z|^2)$ in the complex plane after stereographic projection. On the other hand, the usual hyperbolic metric is given by $|dz|/(1 − |z|^2)$ with the other sign.

Roughly speaking, we then have to consider in the elliptic case domains on the Riemann sphere which contain with every point also the corresponding antipodal point. The conformal mappings have to transform antipodal points onto antipodal points.
Those mappings on the hyperbolic and the elliptic plane are presented in [Kuh]. Several extremal problems are solved there with the Grötzsch strip method or, equivalently, with the Beurling–Ahlfors method of extremal length.

Here we develop a variational method (also of the type “boundary variation”) for the elliptic case which is the complete analogue of the hyperbolic case of Duren and Schiffer [D/Sch]. The main new idea in the construction of the variational formula in the elliptic case is a special conformal welding procedure.

As in the hyperbolic case, this new variational method solves extremal problems in many cases where the method of extremal length is more complicated or ineffective. On the other hand, in several cases the method of extremal length also gives us immediately uniqueness results. Also extremal problems with higher normalizations need additional considerations by using the variational method.

2 Definitions

To make the things more precise, we formulate some definitions in accordance with [Kuh].

**Definition 1.** For the point $z$, let $z^*$ be the antipodal point on the Riemann sphere: $z^* = -1/\bar{z}$.

For a set $\mathcal{M}$, let $\mathcal{M}^*$ be the set of all antipodal points. A set $\mathcal{M}$ is called *diametrically symmetric* if $\mathcal{M}^* = \mathcal{M}$.

A set $\mathcal{M}$ is called *elliptically schlicht* if $\mathcal{M}^* \cap \mathcal{M} = \emptyset$.

In the elliptic case, we have to consider two types of domains, diametrically symmetric domains and elliptically schlicht domains; by way of contrast, in the hyperbolic case we have only one type of domain. (In the interpretation of the elliptic plane as the Riemann sphere with identification of antipodal points, a diametrically symmetric domain is a one-sided or non-orientable domain; an elliptically schlicht domain is a two-sided or orientable domain.)

**Definition 2.** A schlicht conformal mapping $f(z)$ of a diametrically symmetric domain $D$ is called *diametrically symmetric* if $f(z^*) = f(z)^*$ for all $z \in D$.

A schlicht conformal mapping $f(z)$ of an elliptically schlicht domain $D$ is called *elliptically schlicht* if $f(z_1) \neq f(z_2)^*$ for all pairs $z_1, z_2 \in D$. (This means that $f(D)$ is again elliptically schlicht.)

In principle, the solution of extremal problems for elliptically schlicht domains can be deduced by a limit process from the solution for diametrically symmetric domains (cf. [Kuh], p. 12). This corresponds to the fact that an elliptically schlicht