A General First-Order Deformation Solution (*)

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Summary. — The general first-order deformation of representations \( \varphi \) of inhomogeneous Lie algebras \( \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{b} \) into those of semi-simple noncompact Lie algebras \( \mathfrak{g} = \mathfrak{r} + \mathfrak{b} \) is studied. If \( \dim H^1(\mathfrak{g}, \mathfrak{r}) = 1 \) and the irreducible initial representation \( \varphi \) with \( M^2 \neq 0 \) decomposes into a sum of irreducible representations of \( \mathfrak{r} \) with multiplicity 1, then the general solution can be constructed by the aid of a spectral solution of the deformation problem \( \mathfrak{g} \rightarrow \mathfrak{g} \) and the general solution of the corresponding deformation problem \( \mathfrak{g} \rightarrow \mathfrak{g}. \) If \( \mathfrak{g} \) allows transitions within one representation of \( \mathfrak{g} \), the special solution is unique. Otherwise there exists an additional degree of freedom leading to—as three examples studied in the Appendix show—supplementary series of \( \mathfrak{g}. \)

Let \( \mathfrak{g} = \mathfrak{r} + \mathfrak{b} \) \( ([\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r}, [\mathfrak{r}, \mathfrak{b}] \subset \mathfrak{b}; [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}) \) be a finite-dimensional, real, semi-simple Lie algebra contracted (cf. (1)) with respect to the symmetric subalgebra \( \mathfrak{r} \) into the semi-direct sum

\[
\mathfrak{g} \oplus \mathfrak{b} = \mathfrak{r} \oplus \mathfrak{b} \quad ([\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r}, [\mathfrak{r}, \mathfrak{b}] \subset \mathfrak{b}; [\mathfrak{b}, \mathfrak{b}] = 0)
\]

—which differs from \( \mathfrak{g} \) only by the commutability of \( \mathfrak{b} \)—and let \( \varphi \) be an irreducible representation of \( \mathfrak{g} \) on a vector space \( \mathcal{V} \). We look for all first-order deformations of \( \varphi \) leading to representations of \( \mathfrak{g} \), which we have won back from \( \mathfrak{g} \)—as usual—by a second-order deformation of the algebra. We suppose \( \mathfrak{g} \) to be noncompact. (Otherwise we should deform it into \( \mathfrak{g}^* := \mathfrak{r} + i\mathfrak{b} \), the

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noncompact semi-simple dual algebra.) Furthermore, we restrict ourselves to deformations which do not affect the subalgebra \( \mathfrak{g} \), so that we have to deal with the \( \mathfrak{g} \)-relative cohomology, i.e. we look for all 1-cocycles (cf. (3))

\[
(1) \quad \omega \in Z^1(\hat{\mathfrak{g}}, \mathfrak{g}; L(V), \phi')
\]

(the representation \( \phi' : \hat{\mathfrak{g}} \to L(L(V)) \) is defined by

\[
\phi' (\hat{g})(Y) = [\phi(\hat{g}), Y], \quad \hat{g} \in \hat{\mathfrak{g}}, Y \in L(V)
\]

satisfying the subsidiary conditions

\[
(2) \quad \omega(\mathfrak{g}) = 0
\]

and

\[
(3) \quad [\omega(\hat{P}_1), \omega(\hat{P}_2)] = \phi([P_1, P_2])
\]

\( \hat{P}_1, \hat{P}_2 \in \hat{\mathfrak{g}}; P_1, P_2 \) being the corresponding generators of \( \mathfrak{g} \). \( Z^1 \) contains all first-order deformations of \( \phi \) with the algebra \( \hat{\mathfrak{g}} \) remaining fixed. Coboundaries lead to representations of \( \hat{\mathfrak{g}} \) equivalent to \( \phi \) in the first order. The second-order term (3) now selects all those 1-cocycles leading to representations of \( \mathfrak{g} \).

We want to make the following three assumptions:

\begin{enumerate}
\item \( a) \) The considered irreducible representation \( \phi \) belongs to \( M^2 \neq 0 \).
\item An invariant of \( \hat{\mathfrak{g}} \) is always

\[
(4) \quad \hat{I} = G_{\alpha\beta} \hat{P}_\alpha \hat{P}_\beta,
\]

\( G_{\alpha\beta} \) being the Killing form of \( \mathfrak{g} \). Its eigenvalue is the "mass square" \( M^2 \) up to a nonzero factor.
\item \( b) \)

\[
(5) \quad \dim H^1(\hat{\mathfrak{g}}, \mathfrak{g}; L(V), \phi') = 1.
\]

Since \( \omega(\hat{\mathfrak{g}}) = \lambda \cdot \phi(\hat{\mathfrak{g}}) \) always leads to a nonequivalent representation in the case of \( M^2 \neq 0 \), this means

\[
(6) \quad \omega(\hat{\mathfrak{g}}) = \lambda \cdot \phi(\hat{\mathfrak{g}}) + d\omega^{(0)}(\hat{\mathfrak{g}}), \quad \lambda \in \mathbb{C},
\]

\[
(7) \quad \omega(\mathfrak{g}) = d\omega^{(0)}(\mathfrak{g}),
\]

with a 1-coboundary \( d\omega^{(0)} \). \( (\omega^{(0)} \in L(V), d\omega^{(0)}(\hat{g}) = [\phi(\hat{g}), \omega^{(0)}] \).
\end{enumerate}

\[ (2) \quad \text{R. Hermann: Comm. Math. Phys., 2, 251 (1966); 3, 53, 75 (1966); 5, 131, 157 (1967).} \]