On the Determination of Quantum Hamiltonians.

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Summary. — It is shown that, by interpreting the classical equations of motion as operator identities, it is possible to construct quantum Hamiltonians without using classical Hamiltonians.

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Given a quantum Hamiltonian $\hat{H}$, one may determine the probability field through the Schrödinger equation

$$\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

as well as the time evolution of operators through the Heisenberg equation

$$\dot{\hat{A}} = \frac{d\hat{A}}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{A}] + \frac{\partial \hat{A}}{\partial t},$$

The importance of quantum Hamiltonians is obvious and needs not be stressed here. What we shall discuss in this note is the question of how to determine a quantum Hamiltonian corresponding to a given classical system.

In case the classical system is a conservative one where a classical Hamiltonian $H(P_i, x_i)$ can be easily defined, the corresponding quantum Hamiltonian is usually obtained through the standard procedure of making the replacement $p_i \rightarrow (\hbar / i)(\partial / \partial x_i)$. However, if a classical system is defined by the equations of motion rather than by some a priori given Hamiltonians, the determination of classical Hamiltonians for the system is, in general, not an easy matter.
In some cases there may not even exist Hamiltonians, while in some other cases there may be many Hamiltonians corresponding to the same set of equations of motion \(^{(1,2)}\). In fact, for one-dimensional motion, any time-independent constants of motion can be converted into a Hamiltonian \(^{(3)}\). Unfortunately, there does not seem to exist a definite criterion for selecting the correct Hamiltonian, for the purpose of quantization. In view of such difficulties, we set forth to search for some alternative methods that may permit construction of quantum Hamiltonians without having to look for the classical Hamiltonians first. For such a purpose, we simply assume that there exists a quantum Hamiltonian \(\hat{H}\), so that velocity and acceleration operators can be calculated according to (2) as

\[
\hat{x}_i = i\frac{\hbar}{\hbar} [\hat{H}, \hat{x}_i],
\]

\[
\hat{\dot{x}}_i = i\frac{\hbar}{\hbar} [\hat{H}, \hat{\dot{x}}_i] = \left(i\frac{\hbar}{\hbar}\right)[\hat{H}, [\hat{H}, \hat{x}_i]],
\]

where the explicit form of \(\hat{H}\) is yet to be determined. Now, replacing the velocities and accelerations in the classical equations of motion by their operator forms, respectively, and interpreting the original equation of motion as operator identities, one obtains conditions to determine \(\hat{H}\). However, the conditions so obtained are too loose to permit a definite determination of \(\hat{H}\). Thus one may impose additional conditions such as requiring the quantum Hamiltonians to be time-independent second-order linear differential Hermitian operators. With such extra conditions, it becomes possible to construct \(\hat{H}\) explicitly.

We shall illustrate the ideas outlined above by some examples. For simplicity, we limit ourselves to the one-dimensional case. The most general time-independent second-order linear differential operator can be written as

\[
\hat{H}_0 = A \frac{d^2}{dx^2} + B \frac{d}{dx} + C,
\]

where \(A\), \(B\) and \(C\) are some complex functions of \(x\).

Imposing the condition that (5) be Hermitian, one has

\[
\hat{H}_0 = \alpha_1 \frac{d^2}{dx^2} + (\alpha'_1 + i\alpha_2) \frac{d}{dx} + \left(\frac{i}{2} \alpha'_2 + \alpha_3\right),
\]


