Weak convergence of Dirichlet processes*

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Abstract Weak convergence of Markov processes is studied by means of Dirichlet forms and two theorems for weak convergence of Hunt processes on general metric spaces are established. As applications, examples for weak convergence of symmetric or non-symmetric Dirichlet processes on finite and infinite spaces are given.

Keywords: strictly quasi-regular Dirichlet form, weak convergence, Mosco convergence, s.\{\varepsilon^n\}-nest.

It is well known that there is a nice correspondence between Dirichlet forms and Markov processes\cite{1,2}. Recently many authors studied the weak convergence of Markov processes in terms of Dirichlet forms by using different methods. For example, Kuwae and Uemura\cite{1} studied Dirichlet forms ($\varepsilon^n$, $D(\varepsilon^n)$) on $L^2(X, \varphi_n^2 m)$ which admit square field operators:

$$\varepsilon^n(u, v) = \frac{1}{2} \int_X \Gamma_n(u, v) \varphi_n^2 dm, \quad u, v \in D(\varepsilon^n),$$

where $X$ is a locally compact separable metric space. They showed the weak convergence of the symmetric diffusion process sequences associated with the above Dirichlet forms under some assumptions.

Lyons and Zhang\cite{2} considered the following divergence operators:

$$L^n = \frac{1}{2} \sum_{i,j} \partial_j (a^n_{ij} \partial_i) + b^n \cdot \nabla.$$

Let $M_n = \{ \Omega, \mathcal{F}_t, \theta_t, X_t, \{P^n_s : s \in \mathbb{R}^d \} \}$ denote the diffusion process generated by $L^n$, where $\Omega = C([0, \infty), \mathbb{R}^d)$. They proved that $M_n$ converges weakly when the corresponding coefficients converge.

Röckner et al.\cite{3} proved that the diffusion processes on Hilbert space $E$ associated with Dirichlet forms of type:

$$\delta(u, v) = \int_E \langle A(z) \nabla u(z), \nabla v(z) \rangle_H \mu(dz)$$

can be approximated by finite-dimensional processes, where $H$ is a Hilbert space and $H \rightarrow E$ is Hilbert-Schmidt embedding.

In this paper we generalize the method used by Ma, Röckner and Zhang\cite{3} who proved that arbitrary Dirichlet process can be approximated by Markov chains. Combining the compactification method with the characterization theorem for relative compactness of processes\cite{4}, we obtain

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two theorems (Theorems 1 and 2) on weak convergence of symmetric and non-symmetric Dirichlet processes on general metric spaces (including finite and infinite spaces).

Theorems 1 and 2 will be given in sec. 2. Several examples of the applications of the above two theorems are given in sec. 3. In example 1 we improve a result of Kuwae and Uemura\(^{1}\) by removing the restricted condition that the topology induced by the pseudo-metric defined by Dirichlet form must be coincident with the original topology of the state space. With example 2 we prove that arbitrary Dirichlet process can be weakly approximated in the original topology by the compound Poisson processes associated with their Yosida approximations. In example 3 we obtain the weak convergence of divergence operators (studied by Lyons and Zhang) without using the Girsanov transformation. In example 4 we show that the diffusion processes associated with the following Dirichlet form on separable Banach space \(E\):

\[
\varepsilon(u, v) = \sum_{i=1}^{\infty} \frac{1}{k_i} \frac{\partial u}{\partial i} \frac{\partial v}{\partial i} d\mu, u, v \in \mathcal{F}_{\infty}
\]

can be approximated in a way of finite dimension.

In addition to the examples given in sec. 3, Theorems 1 and Theorem 2 have also other applications. It is well known that Lyons-Zheng decomposition\(^5\) is a powerful tool for studying the tightness of processes\(^3,6\). But it is only applicable to conservative diffusion processes. Theorems 1 and 2 can be used to treat non-conservative processes and processes with jump parts. The details will be studied in a forthcoming paper.

1 Main theorems

For the terminologies and notations used in this paper refer to [2].

Let \(E\) be a metric space and let \(m\) be a \(\sigma\)-finite measure on its Borel \(\sigma\)-algebra \(\mathcal{B}(E)\). Let \(\{\varepsilon^n, D(\varepsilon^n)\}_{n \in \mathbb{N}}\) be a sequence of strictly quasi-regular Dirichlet forms on \(L^2(E, m)\) (for the definition see definition V. 2. 11 of ref. [2]) with associated semigroups \(\{(T^n_t)_{t \geq 0}\}_{n \in \mathbb{N}}\), resolvents \(\{(G^n_u)_{u > 0}\}_{n \in \mathbb{N}}\), generators \(\{L^n\}_{n \in \mathbb{N}}\) and co-associated semigroups \(\{(T^n_t)_{t \geq 0}\}_{n \in \mathbb{N}}\), co-resolvents \(\{(G^n_u)_{u > 0}\}_{n \in \mathbb{N}}\) and co-generators \(\{L^n\}_{n \in \mathbb{N}}\), respectively.

Fix arbitrary \(\varphi \in \mathcal{B}_b(E), \varphi > 0\) m. a. e. and \(\int_E \varphi d\mu = 1\). Let \(S^n_f\) denote the family of all 1-excessive functions in \(D(\varepsilon^n)\). Following ref. [2], V. 2, for any open set \(U \subset E\), we define

\[
\text{Cap}_{1, \varepsilon^n}(U) = \sup \left\{ \text{Cap}_{\varepsilon^n}(U) \mid u \in S^n_f, u \leq 1 \right\},
\]

(for the definition of \(\text{Cap}_{1, \varepsilon^n}\) refer to definition III. 2. 4 of ref. [2]), and for arbitrary \(A \subset E\), define

\[
\text{Cap}_{1, \varepsilon^n}(A) = \inf \left\{ \text{Cap}_{1, \varepsilon^n}(U) \mid A \subset U \subset E, U \text{ is open} \right\}.
\]

Definition 1. An increasing sequence \(\{F_k\}_{k \in \mathbb{N}}\) of closed sets of \(E\) is called an s.\(\{\varepsilon^n\}\)-nest (strictly \(\{\varepsilon^n\}\)-nest) if

\[
\limsup_{k \to \infty} \text{Cap}_{1, \varepsilon^n}(F_k) = 0.
\]

1) See footnote 1) on page 8.