The Hausdorff measure of the self-similar sets*

— The Koch curve

ZHOU Zuoling (周作陵)
(Linnan College, Zhongshan University, Guangzhou 510275, China)

Received October 25, 1997

Abstract The self-similar sets satisfying the open condition have been studied. An estimation of fractal, by the definition can only give the upper limit of its Hausdorff measure. So to judge if such an upper limit is its exact value or not is important. A negative criterion has been given. As a consequence, the Marion's conjecture on the Hausdorff measure of the Koch curve has been proved invalid.

Keywords: self-similar sets, Hausdorff measure and dimension, Koch curve.

In the investigation of the fractal geometry, to count the Hausdorff dimension of fractals is very difficult and to count the Hausdorff measure is more difficult. Up to now, the fractals studied most successfully are the self-similar sets satisfying the open set condition. For such a fractal, the Hausdorff dimension equals its self-similar dimension and the Hausdorff measure is a positive finite number. But, for such a fractal, to count its Hausdorff measure is also very difficult. We have discovered that using the definition to estimate its Hausdorff measure, we can only obtain its upper limit. Of course, the best upper limit is its exact value, so to judge if such an upper limit is its exact value or not is important. In this paper, we give a negative criterion, and as a consequence, negate the Marion's conjecture on the Hausdorff measure of the Koch curve.

1 Self-similar sets

For some definitions, notations and known results, refer to reference [1].

Let $D \subseteq \mathbb{R}^n$ be closed, $n \geq 1$. Denote by $|x - y|$ the Euclidean distance between $x, y \in \mathbb{R}^n$ and by $|E|$ the diameter of the subset $E \subseteq \mathbb{R}^n$. A map $S: D \to D$ is called a contraction on $D$ if there is a number $c$ with $0 < c < 1$ such that

$$|S(x) - S(y)| \leq c|x - y|, \forall x, y \in D.$$ 

$S$ is called a similarity and $c$ is called similar ratio if the equality holds, i.e. $|S(x) - S(y)| = c|x - y|$. It is easy to see that a similarity geometrically similarly maps any subset of $D$ into its image with similar ratio $c$.

Let $S_i$ be a similarity with ratio $c_i$, $i = 1, 2, \cdots, m$, $m \geq 1$. There is a unique $F \subseteq D$ such that $F = \bigcup_{i=1}^{m} S_i(F)$. $F$ is called the self-similar set determined by $\{|S_i|\}$. We say that $F$ satisfies the open set condition if there is a nonempty open set $U \subseteq \mathbb{R}^n$ such that $\bigcup_{i=1}^{m} S_i(U) \subseteq U$ and $S_i(U) \cap S_j(U) = \emptyset$, $0 < i < j \leq m$.

If $F$ satisfies the open set condition, then

* Project partially supported by the State Scientific Commission and the State Education Commission.
where \( s \) satisfies \( \sum_{s} c_i^s = 1 \), and both \( \dim_H(F) \) and \( H'(F) \) denote the Hausdorff dimension and \( s \)-dimensional Hausdorff measure of \( F \), respectively. \( s \) also is called similar dimension of \( F \). We denote \( s = \dim_H(F) \) in this section.

Let \( a = \{ U_i, \ l \geq 0 \} \) be a \( \delta \)-covering \( \delta > 0 \) of \( F \) with \( \delta > 0 \). It is easy to see that \( \{ S_i(U_l), \ l \geq 0 \} \) is a \( c_i \delta \)-covering of \( S_i(F), i = 1, 2, \ldots, m \). Similarly, 
\[
| S_{i_1} \cdots S_{i_k}(U_l), \ l \geq 0 |
\]
is a \( c_{i_1} \cdots c_{i_k} \delta \)-covering of 
\[
S_{i_1} \cdots S_{i_k}(F), \ 1 \leq i_1, \ldots, i_k \leq m, \ k > 0.
\]
This covering is denoted by \( a_{i_1} \cdots i_k \). Obviously 
\[
| S_{i_1} \cdots S_{i_k}(U_l) | = c_{i_1} \cdots c_{i_k} | U_l |, \ \forall \ l \geq 0.
\]

Denote by \( J_k \) the set of all \( k \)-sequences \((i_1, \ldots, i_k), 1 \leq i_1, \ldots, i_k \leq m, \ k \geq 1, \) and put \( F_{i_1, \ldots, i_k} = S_{i_1} \cdots S_{i_k}(F) \).

It is not hard to see that \( F = \bigcup_{k} F_{i_1, \ldots, i_k} \). Hence, when \((i_1, \ldots, i_k)\) runs over \( J_k \), \( | a_{i_1} \cdots i_k | \) forms a \( \delta_k \)-covering of \( F \), where 
\[
\delta_k = \max_{J_k} | c_{i_1} \cdots c_{i_k} \delta | > 0.
\]
This covering is called similar compress of \( a \) and denoted by 
\[
| a_{i_1} \cdots i_k |_{J_k}.
\]

**Proposition 1.** Let \( \delta > 0 \). Then \( H'(F) = H_\delta(F) \).

**Proof.** Let \( a = \{ U_i, \ l \geq 0 \} \) be a \( \delta \)-covering of \( F \). According to the definition, we have 
\[
H_\delta(F) \leq \sum_{l=0}^{\infty} | U_l |^\delta.
\]
As similar compress \( | a_{i_1} \cdots i_k |_{J_k} \) of \( a \) is a \( \delta_k \)-covering of \( F \), we get 
\[
H_\delta(F) \leq \sum_{l=0}^{\infty} \sum_{(i_1, \ldots, i_k) \in J_k} c_{i_1}^\delta \cdots c_{i_k}^\delta | U_l | = \sum_{l=0}^{\infty} \sum_{i_1, \ldots, i_k} c_{i_1}^\delta \cdots c_{i_k}^\delta | U_l |
\]
\[
= \sum_{l=0}^{\infty} \left( \sum_{i_1} c_{i_1}^\delta \right) \cdots \left( \sum_{i_k} c_{i_k}^\delta \right) | U_l |^\delta = \sum_{l=0}^{\infty} | U_l |^\delta.
\]
When \( k \to \infty \), there holds \( \delta_k \to 0 \). By the definition, we get 
\[
H'(F) \leq \sum_{l=0}^{\infty} | U_l |^\delta,
\]
so 
\[
H'(F) \leq H_\delta(F).
\]
On the other hand, it is clear that \( H'(F) \geq H_\delta(F) \).

**Corollary.** Let \( a = \{ U_i, \ l \geq 0 \} \) be any covering of \( F \). Then 
\[
H'(F) \leq \sum_{l=0}^{\infty} | U_l |^\delta.
\]
By Proposition 1, it is clear.

Let \( a = \{ U_i, \ l \geq 0 \} \). We say that \( a \) is an \( H' \)-a.e. covering of \( F \) if \( F - \bigcup_{l=0}^{\infty} U_l \) is an \( H' \)-zero