Noether's Theorem and Ermakov Systems for Nonlinear Equations of Motion.

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**Summary.** — We present a simple but important generalization of the use of Noether's theorem to generate invariants for time-dependent nonlinear equations of motion. Our main result is that Noether's theorem produces invariants even when the equation of motion contains arbitrary functions. The results obtained by using Noether's theorem are the most general results for which one obtains explicit nonlinear superposition laws for the class of equations of motion considered. We also correct a recent paper on this same subject.

I. — Introduction.

Ray and Reid (1) have recently proven that the quantity

\[ I = \frac{1}{2} (\dot{\varphi} \ddot{x} - x \dot{\varphi})^2 + \int f(\eta) d\eta + \int g(\eta) d\eta \]

is constant if \( \varphi \) satisfies the equation of motion

\[ \ddot{\varphi} + \omega^2(t) \varphi = f(x/\varphi)/\varphi^2 x \]

and \( x \) satisfies the auxiliary equation

\[ \ddot{x} + \omega^2(t) x = g(x/\varphi)/x^2 \varphi , \]

where $\omega^2(t)$, $f(x|q)$ and $g(q|x)$ are arbitrary functions. We refer to (2) and (3) as an Ermakov pair of equations and to (1) as the Ermakov invariant associated with this pair. The derivation of (1) can be carried out most directly by eliminating $\omega^2(t)$ between (2) and (3) and performing some simple manipulations to arrive at (1).

Several special cases of the Ermakov system (1)-(3) have proven useful in discussing time-dependent systems. The Lewis invariant, for which $f = 0$, $g = q|x$ is the best-known example of such a system. The $q$ equation in this case is the time-dependent harmonic-oscillator equation

$$(4) \quad \ddot{q} + \omega^2(t)q = 0$$

which arises in many physical problems. The auxiliary equation in this case is the Ermakov-Pinney equation

$$(5) \quad \ddot{x} + \omega^2(t)x = 1/x^3.$$ 

The Lewis invariant has the form

$$(6) \quad I = \frac{1}{2} [(\dot{q}\dot{x} - x\dot{q})^2 + (q|x)^2].$$

The Lewis invariant was used by Lewis and Riesenfeld (2) to give an exact quantum-mechanical treatment of the time-dependent harmonic oscillator.

The elimination of $\omega^2$ derivation of $I$ given above has the weakness that one must know both of eqs. (2) and (3) in order to derive $I$. If one is interested in extensions to other equations of motion not having form (2), how does one find the auxiliary equation (3)? Noether's theorem furnishes a method of determining both the invariant and an auxiliary equation. There have been other methods of attaching this same problem which have been discussed by Ray and Reid (4).

Katzin and Levine (5) and Lutzky (6) derived both the Lewis invariant and the auxiliary equation (5) by applying Noether's theorem to the Lagrangian for (4), namely

$$(7) \quad L = \frac{1}{2} (\dot{q}^2 - \omega^2(t)q^2).$$

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