LINEAR COMBINATIONS OF BERNSTEIN OPERATORS ON A SIMPLEX *

Wu Zhengchang (吴正昌)
(Zhejiang University, China)

Received Feb. 18, 1990

Abstract

In this paper, we extend the idea of linear combinations of Bernstein polynomials to multidimensional case.

1. Introduction

The multidimensional Bernstein polynomials were introduced early in [6]. It is only recent they have drawn much more attention ([2] [3] [7] [8]). In addition to their theoretical interest, for partial reason, it is that they have been successfully used in CAGD to generate satisfying surfaces. However, as approximation tool it is known that the approximation order of Bernstein polynomials on a simplex is limited \((O(\frac{1}{n}))^n\). In order to obtain faster convergence one can consider linear combinations of Bernstein polynomials which was first studied by P.L. Butzer in univariate case and later was deeply studied (for example [4] [5]). In this paper we shall extend the idea to multidimensional case, it is more complicated than that of one dimension. In § 2 we introduce and study polynomials

\[ T_{a,x} = B_n((\cdot - x)^a)(x), \]

which will play an important role in investigating Bernstein polynomials. Then in § 3 basing on the properties of the polynomial \(T_{a,x}\) we obtain the asymptotic expansion of Bernstein polynomials, which have independent interest. Finally in § 4 we define the combinations of Bernstein operators on a simplex and study their approximation properties.

* This paper is the part of the Ph.D. dissertation of author when he studied in the Graduate School of Zhejiang University.
2. Polynomial $T_{n,s}$

First we recall some notations.

$\mathbb{R}^m$ is real Euclidean space. $e_1, e_2, \ldots, e_m$ are canonical basis of $\mathbb{R}^m$. Let

$x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, $y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$, as usual, $x \cdot y = \sum_{i=1}^{m} x_i y_i$, $\|x\| = (x \cdot x)^{1/2}$. For a function $f(x)$, the directional derivative in direction $y$ is defined by

$$(D_y f)(x) = \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}.$$  \hspace{1cm} (2.1)

Let

$$D_i := D_{e_i}, \quad D^\mu := D_1^{\mu_1} D_2^{\mu_2} \cdots D_m^{\mu_m},$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{Z}_+^m$.

Let $v^1, v^2, \ldots, v^{m+1}$ be affinely independent in $\mathbb{R}^m$, $\sigma$ denote a simplex with vertices $v^1, v^2, \ldots, v^{m+1}$. For $x = (x_1, x_2, \ldots, x_m) \in \sigma$, there exists $m+1$-tuple $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{m+1})$ such that

$$x = \sum_{i=1}^{m+1} \zeta_i v^i, \quad \sum_{i=1}^{m+1} \zeta_i = 1, \quad \zeta_i \geq 0, \quad i = 1, 2, \ldots, m + 1.$$  

The $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{m+1})$ is called barycentric coordinate of $x$ with respect to $\sigma$. Geometrically $x$ and $\zeta$ express the same point. If $f(x)$ is defined on $\sigma$, then its corresponding Bernstein polynomial is defined as follows

$$B_\alpha f(x) = \sum_{\mu=\alpha} B_\mu(x) f(x).$$

Here

$$B_\alpha(x) = \left( \frac{|\alpha|}{\alpha!} \right) \zeta^\alpha,$$

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{m+1}) \in \mathbb{Z}_+^{m+1}, \quad \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{m+1}), \quad \zeta^\alpha = \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \cdots \zeta_{m+1}^{\alpha_{m+1}},$$

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{m+1}, \quad \left( \frac{|\alpha|}{\alpha!} \right) = \frac{|\alpha|!}{\alpha!}, \quad \alpha! = \alpha_1! \alpha_2! \cdots \alpha_{m+1}!;$$

$$x_\alpha = \sum_{i=1}^{m+1} \frac{\alpha_i}{\alpha!} v^i, \quad \text{for } |\alpha| = n.$$  

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{Z}_+^m$, $x = (x_1, x_2, \ldots, x_m) \in \sigma$, we define