On Certain Physical-Region Singularities in \( S \)-Matrix Theory (*)

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(ricevuto il 17 Novembre 1965; manoscritto revisionato ricevuto il 28 Marzo 1966)

Summary. — Discontinuities of the four-to-four particle amplitude across the cuts in the physical region associated with two quadrilateral singularities (Fig. 1 and 2) are evaluated by \( S \)-matrix methods. The Cutkosky formula and positive-\( \alpha \) criterion for these singularities are deduced.

1. — Introduction.

Investigations of the singularity structure of \( S \)-matrix amplitudes are important because of the calculative methods to which they might be expected to lead, and because they may shed light on properties of general theoretic interest (1).

It is already clear that «cluster decomposition», unitarity and weak analyticity—that is, the requirement that the connected parts of the amplitude be boundary values of analytic functions—exercise together a very powerful control over the singularities in the physical region. A recent paper by Landshoff and Olive (2) makes it plausible that this control may even be complete.

(*) The research reported in this document has been sponsored in part by the Air Force Office of Scientific Research, OAR, under Grant AF EOAR 63-79 with the European Office of Aerospace Research, United States Air Force.


The form the unitarity equations take when amplitudes are split into their connected parts suggests a fruitful analogy with the Feynman diagrams of perturbation theory. Feynman integrals were shown by Landau (3) to have singularities which must lie on «Landau curves» defined by

\[ \alpha_i = 0 \quad \text{or} \quad q_i^2 = m_i^2 \quad \text{for each internal line}, \]

and

\[ \sum \alpha_i q_i = 0 \quad \text{round any loop}. \]

Further, when a negative imaginary part is attached to each internal mass it is only in that region of the Landau curve which corresponds to all the \( \alpha \)'s being positive that the integral has such a singularity (2). Polkinghorne (4) has shown that the singularities of complete connected amplitudes generated by unitarity and crossing do lie on the Landau curves, and it is a strong conjecture that the physical amplitudes are only singular in the positive-\( \alpha \) regions, as is suggested by the summing of an infinite series of perturbation-theory graphs. A rigorous proof of this property in general is so far lacking, but in their paper Landshoff and Olive were able to show that it does follow for the triangle singularity (5) of the three-to-three particle amplitude from the rather weak set of assumptions at the head of the paragraph, in particular without recourse to crossing or Hermitian analyticity. If this result can be generalized, the singularities of the physical region will have been proved to be uniquely determined by unitarity and weak analyticity. It may also be a way of making the supposed link between causality and analyticity, as on the Coleman-Norton interpretation (6) the positive-\( \alpha \) condition represents causality: the amplitudes are then only singular when there can be a physical intermediate scattering process, «forward in time». Finally, it would confirm the existence of a «hierarchical structure» of the kind recognized in perturbation theory, where singularities are switched on or off at points at which their Landau curve touches the curve of the lower singularity obtained by setting an \( \alpha \) equal to zero (equivalent to contracting an internal line) (7).

Normal thresholds, being the «lowest» singularities in the hierarchy, are not switched off in this way. It is a feature of normal thresholds that the unitarity equation takes different forms above and below threshold; this feature is