MULTI-SAMPLE SCALE PROBLEM:
UNKNOWN LOCATION PARAMETERS*)

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(Received Jan. 13, 1967)

Summary

In continuation of the author’s paper [3], here a class of rank order tests for the homogeneity of scale parameters is proposed and their properties are studied when the location parameters are unknown.

1. Introduction

Let \( X_{j1}, \ldots, X_{jm_j}, j=1, \ldots, c \) be independent and identically distributed random variables from populations with continuous cumulative distribution functions \( F_j(x) = F((x-x_j)/\sigma_j), \ j=1, \ldots, c. \) \( x_j \) and \( \sigma_j \) are the location and the scale parameters respectively for the distribution function \( F_j(x). \) We are interested in testing the hypothesis \( H_0: \sigma_1 = \cdots = \sigma_c \) against \( H_1: \sup \{ \sigma_1 - \sigma_j \} > 0. \)

Let \( \hat{\xi}_j = \hat{\xi}_j(X_{j1}, \ldots, X_{jm_j}) \) be a consistent estimator of \( x_j \) such that \( N^{1/2}(\hat{\xi}_j - \xi_j) \) (where \( N = \sum m_j \)) is bounded in probability for each \( j=1, \ldots, c. \) Consider now the modified sample \( \bar{X}_{jr} = X_{jr} - \hat{\xi}_j, \ r=1, \ldots, m_j; \ j=1, \ldots, c, \) and denote \( Z_{j,r}^{\xi_j} = 1 \) if the \( r \)th smallest observation from the combined sample \( X_{jr}^*, r=1, \ldots, m_j; \ j=1, \ldots, c, \) is from the \( j \)th set and otherwise \( Z_{j,r}^{\xi_j} = 0. \) Then we propose to consider the test statistic \( V^*(W) \) defined as

\[
(1.1) \quad V^*(W) = \sum_{j=1}^{c} m_j [(T_{N,j}^{*} - \mu_{N,j})/A_N]^2
\]

where

\[
(1.2) \quad T_{N,j}^{*} = \sum_{i=1}^{N} E_{r}[V^{(r)}] Z_{N,i}^{\xi_j}/m_j.
\]

*) This paper represents the results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the Office of Naval Research, Contract Nonr-285(38). Reproduction in whole or in part is permitted for any purpose of the United States Government.
Here $V^{(1)}<\cdots<V^{(N)}$ is an ordered sample of size $N$ from a distribution $\mathcal{F}$. $E$ denotes the expectation and $x_{\nu}$ and $A_{v}$ are normalizing constants to be defined below. The motivation behind the use of $T^*$'s is to extend a class of tests of homogeneity of scale parameters proposed and studied by the author [3], for the case when the locations are completely unknown. The test consists in rejecting $H_0$ at a significance level $\alpha$ if $L^*_0(\mathcal{F})$ exceeds some predetermined number $L^*_{\alpha}$. From a theorem proved in the next section, it follows that when $H_0$ is true, $L^*(\mathcal{F})$ is asymptotically distributed as a chi-square with $c-1$ degrees of freedom. Thus a large sample approximation for $L^*_0(\mathcal{F})$ is provided by the upper $\alpha$-point of the chi-square distribution with $c-1$ degrees of freedom. References to prior work on the two sample and the c-sample nonparametric tests for scale may be found in Raghavachari [5], Sen [6] and the author [3].

Finally, we may mention that in [3] the author also considered multisample versions of the two-sample scale tests due to Ansari and Bradley, and Mood respectively for the case when the location parameters are known. Essentially the same techniques as in the present paper can be used to establish that the asymptotic distributions of the Ansari-Bradley, and the Mood statistics are the same whether the locations are known or unknown. However, for the sake of brevity we restrict ourselves to the $L^*(\mathcal{F})$ test. The asymptotic distribution freeness of the Mood test when the locations are unknown is also established by Crouse [1]. For the problem of obtaining confidence intervals for scale parameters, the reader is referred to Sen [7].

2. Assumptions and notations

Let $X_{j1}<\cdots<X_{jm_j}$, $j=1,\ldots,c$, be the ordered observations of a random sample from a population having absolutely continuous cdf $F_j(x-x_j)$, where $x_j$ is the median of $F_j(x)$, $j=1,\ldots,c$. Let $\lambda_j=m_j/N$ and assume that for all $N$, the inequalities $0<\lambda_0=\cdots=\lambda_c<1-\lambda_0<1$ hold for some fixed $\lambda_0\leq 1/c$.

Denote

\begin{align}
S^{(j)}_{\nu}(x) &= m_j^{-1} \quad \text{(number of } X_{jr} \text{ such that } X_{jr}-x_j \leq x, \\
n_{r=1,\ldots,m_j},
\end{align}

\begin{align}
H_N(x) &= \sum_{j=1}^{c} \lambda_j S^{(j)}(x).
\end{align}

Let $\hat{\xi}_j=\hat{\xi}_j(X_{j1},\ldots,X_{jm_j})$ be a consistent estimator of $\xi_j$ such that $N^{1/2} (\hat{\xi}_j-\xi_j)$ is bounded in probability. Define

\begin{align}
S^{(j)}_{\nu}(x) &= m_j^{-1} \quad \text{(number of } X_{jr} \text{ such that } X_{jr} \leq x, \\
n_{r=1,\ldots,m_j},
\end{align}