AN OPTIMAL THEOREM FOR THE SPHERICAL MAXIMAL OPERATOR ON THE HEISENBERG GROUP

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ABSTRACT

Let $\Gamma^n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group and $\mu_r$ be the normalized surface measure on the sphere of radius $r$ in $\mathbb{C}^n$. Let $Mf = \sup_{r>0} |f \ast \mu_r|$. We prove an optimal $L^p$-boundedness result for the spherical maximal function $Mf$, namely we prove that $M$ is bounded on $L^p(\Gamma^n)$ if and only if $p > \frac{2n}{2n-1}$.

1. The main results

Let $H^n = \mathbb{C}^n \times \mathbb{R}$ be the $(2n + 1)$-dimensional Heisenberg group with the group law

$$(z, t)(w, s) = \left(z + w, t + s + \frac{1}{2} \text{Im} z \cdot \bar{w}\right).$$

Given a function $f$ on $H^n$ consider the spherical means

$$(1.1) \quad f \ast \sigma_r(z, t) = \int_{|w|=r} f(z - w, t - \frac{1}{2} \text{Im} z \cdot \bar{w}) d\sigma_r(w)$$

where $\sigma_r$ is the normalised surface measure on the sphere $S_r = \{(z, 0) : |z| = r\}$ in $H^n$. In [2] Nevo and the second author studied the maximal and pointwise
ergodic theorems in $L^p$ for these spherical means. They showed that $\{\sigma_r\}$ is
a pointwise ergodic family in $L^p$ for every ergodic action of $H^n$, $n \geq 2$ on a
probability space (see [2] for the relevant definitions) for all $p > (2n-1)/(2n-2)$. For
actions of the reduced Heisenberg group the range of $p$ was extended to $p > 2n/(2n-1)$ and it was conjectured that the same should be true for the
full group $H^n$.

A basic ingredient in the proof of the ergodic theorem is the $L^p$ boundedness of the maximal function

\[(1.2) \quad M_{\sigma} f(z,t) = \sup_{r>0} |f * \sigma_r(z,t)|\]

associated to the spherical means. In [2] it was shown that $M_{\sigma}$ is bounded on
$L^p(H^n)$ for all $p > (2n-1)/(2n-2)$. In this paper we establish the optimal result, namely

**Theorem 1.1:** Let $n \geq 2$. Then the maximal operator defined by (1.2) is bounded on $L^p(H^n)$ if and only if $p > 2n/(2n-1)$.

This is the analogue of the celebrated spherical maximal theorem of Stein [3]
on $\mathbb{R}^n$. Note that the spherical means $\sigma_r$ are averages over the sphere $S_r$ which is of co-dimension 2 and hence more singular. As a consequence of Theorem 1.1 we obtain the following result.

**Theorem 1.2:** Let $n \geq 2$. Then $\{\sigma_r\}$ is a pointwise ergodic family in $L^p$ for all $p > 2n/(2n-1)$.

The following remarks are in order. The main result, namely Theorem 1.1, has been recently proved by Müller and Seeger [1] using different methods. Actually, they have extended the above result to a more general setting of surfaces and to a class of step two nilpotent groups including all $H$-type groups. They use Fourier integral operators to study the maximal function whereas we prove Theorem 1.1 by modifying the arguments presented in [2] and combining it with Stein’s original square function method. Our proof of Theorem 1.1 is also valid in the general set-up of $H$-type groups. Though the results in [1] are more general, our approach may be of independent interest as it does not appeal to the theory of Fourier integral operators.

The spherical means $f * \sigma_r$ are naturally associated to the Gelfand pair $(H^n,U(n))$ where $U(n)$ is the unitary group. For $K$-spherical means associated to other Gelfand pairs $(H^n,K)$ and the associated maximal functions we refer to [7].