Two-Dimensional Stressed State of an Anisotropic Body with Holes, Elastic Inclusions, and Cracks

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Generalized complex potentials are used to solve a problem for the elastic stressed state of a body with a general rectilinear anisotropy in a two-dimensional stressed-deformed state. The body has longitudinal cavities, inclusions, planar cracks, or rigid lamellar inclusions. A least-squares method is used to reduce the problem to a system of linear algebraic equations which is solved for the unknown constants in the complex potentials.

A method has been proposed \([1, 2]\) for determining a two-dimensional stressed state and stress intensity factors for anisotropic bodies with arbitrary cavities and cracks. In this article that method is extended to the case in which the body has holes and elastic inclusions, as well as planar cracks and absolutely rigid inclusions.

Let us consider a cylindrical body with a general rectilinear anisotropy acted on by external forces in a two-dimensional stressed state. Let the body be weakened by \(L_0\) longitudinal cavities with elliptical cross sections and whose surfaces can be tangent, intersect, and form cavities with complicated shapes or a surface of planar cuts or absolutely rigid lamellar inclusions. Let the surfaces of \(J\) cavities be free from support (main problem I) or rigidly supported (main problem II). Cylindrical elastic inclusions of another anisotropic material are sealed or cemented into the remaining \(L-J\) cavities without prestressing. In this way, a multiply connected region \(S\) with a complicated configuration is formed in the transverse cross section of the body: this region is bounded by the external contour \(L_o\) and the ellipses \(L_i (i = 1, L)\) which can be positioned arbitrarily with respect to one another, while some of them can be transformed into rectilinear cuts. We denote the finite regions corresponding to the elastic inclusions by \(S_i (i = I+1, L)\). In the case where \(L_0\) goes to infinity, we obtain an infinite body with cylindrical cavities. External forces act on the cylindrical surfaces and, in the case of an infinite body, also at infinity in the form of the stress constants \(\sigma_0\), \(\alpha_0\), \(\alpha_y\), and \(\alpha_{xy}\). There is no rotation \(\omega^\circ\) at infinity.

Let us choose local coordinate systems \(O_{1x1y}\) in the plane of a transverse cross section with their origins at the centers of \(L_i\) and directed along the semiaxes \(a_i\) and \(b_i\) so that in these systems the parametric equations for the ellipses have the form

\[
x_i = a_i \cos \theta; \quad y_i = b_i \sin \theta.
\]

while in the system \(Oxy\) they have the form

\[
x = x_{ol} + x_i \cos \varphi_i - y_i \sin \varphi_i; \quad y = y_{ol} + x_i \sin \varphi_i + y_i \cos \varphi_i,
\]

where \(x_{ol}\) and \(y_{ol}\) are the coordinates of the center \(L_o\) in the \(Oxy\) system; \(\varphi_i\) is the angle between the positive directions of the \(Ox\) and \(Oy\) axes; and \(\theta\) is a parameter which ranges from 0 to \(2\pi\).

A determination of the stressed state of this composite body reduces to finding the complex potentials \(\Phi_k(z_k) (k = 1, 3)\) for the body and \(\Phi_k^{(i)}(z_{ki}^{(i)}) (i = 1, 3; j = I+1, L)\) for the inclusions from the appropriate boundary conditions at the contours \(L_i\). Once the complex potentials have been determined, the stresses and strains in the body and in the \(i\)th inclusions are calculated using the formulas \([2, 3]\).

\[
\left( \sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz} \right) = 2\Re \sum_{k=1}^{3} \left( \lambda_{1k}, \lambda_{2k}, \lambda_{4k}, \lambda_{5k}, \lambda_{6k} \right) \Phi_k(z_k),
\]
\[
\sigma_z = -\left( a_{13}\sigma_x + a_{23}\sigma_y + a_{34}\tau_{xy} + a_{35}\tau_{xz} + a_{36}\tau_{yz} \right)/\sigma_{33};
\]
\[
\left( u, v, w \right) = 2\Re \sum_{k=1}^{3} \left( p_k, q_k, r_k \right) \Phi_k(z_k) + \left( -\omega_3y + u_0, \omega_3x + v_0, w_0 \right);
\]
\[
\left( \sigma_x^{(i)}, \sigma_y^{(i)}, \tau_{xy}^{(i)}, \tau_{xz}^{(i)}, \tau_{yz}^{(i)} \right) = 2\Re \sum_{k=1}^{3} \left( \lambda_{1k}^{(i)}, \lambda_{2k}^{(i)}, \lambda_{4k}^{(i)}, \lambda_{5k}^{(i)}, \lambda_{6k}^{(i)} \right) \Phi_k^{(i)}(z_k^{(i)}),
\]
\[
\sigma_z^{(i)} = -\left( a_{13}^{(i)}\sigma_x^{(i)} + a_{23}^{(i)}\sigma_y^{(i)} + a_{34}^{(i)}\tau_{xy}^{(i)} + a_{35}^{(i)}\tau_{xz}^{(i)} + a_{36}^{(i)}\tau_{yz}^{(i)} \right)/\sigma_{33}^{(i)};
\]
\[
\left( u^{(i)}, v^{(i)}, w^{(i)} \right) = 2\Re \sum_{k=1}^{3} \left( p_k^{(i)}, q_k^{(i)}, r_k^{(i)} \right) \Phi_k^{(i)}(z_k^{(i)});
\]
\[
\left( u^{(i)}, v^{(i)}, w^{(i)} \right) = 2\Re \sum_{k=1}^{3} \left( p_k^{(i)}, q_k^{(i)}, r_k^{(i)} \right) \Phi_k^{(i)}(z_k^{(i)}) + \left( -\omega_3y + u_0^{(i)}, \omega_3x + v_0^{(i)}, w_0^{(i)} \right).
\]

Here

\[
\begin{align*}
\lambda_{1j} & = \mu_j^2; \quad \lambda_{2j} = \mu_j; \quad \lambda_{3j} = -1; \quad \lambda_{4j} = \lambda_{5j} = \mu_j; \quad \lambda_{6j} = -\mu_j; \quad (j = 1, 2); \\
\lambda_{13} & = \mu_3^2; \quad \lambda_{23} = \lambda_3; \quad \lambda_{43} = \mu_3; \quad \lambda_{53} = -\mu_3; \quad \lambda_{63} = \mu_3; \\
\lambda_{14} & = \mu_4^2; \quad \lambda_{24} = \mu_4; \quad \lambda_{34} = -1; \quad \lambda_{44} = \lambda_{54} = \mu_4; \quad \lambda_{64} = -\mu_4; \\
\lambda_{15} & = \mu_5^2; \quad \lambda_{25} = \lambda_5; \quad \lambda_{35} = -1; \quad \lambda_{45} = \mu_5; \quad \lambda_{55} = -\mu_5; \quad \lambda_{65} = \mu_5; \\
\lambda_{16} & = \mu_6^2; \quad \lambda_{26} = \lambda_6; \quad \lambda_{36} = -1; \quad \lambda_{46} = \mu_6; \quad \lambda_{56} = -\mu_6; \quad \lambda_{66} = \mu_6.
\end{align*}
\]

\(\mu_k\) and \(\mu_k^{(i)}\) are, respectively, the roots of the characteristic equations

\[
l_4(\mu)/l_2(\mu) - l_3^2(\mu) = 0;
\]

and

\[
l_4^{(i)}(\mu)/l_2^{(i)}(\mu) - l_3^{(i)}(\mu) = 0;
\]

\[
l_4(\mu) = \beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12} + \beta_{66})\mu^2 - 2\beta_{26}\mu + \beta_{22},
\]

\[
l_3(\mu) = \beta_{15}\mu^3 - (\beta_{14} + \beta_{56})\mu^2 + (\beta_{25} + \beta_{46})\mu - \beta_{24},
\]

\[
l_2(\mu) = \beta_{55}\mu^2 - 2\beta_{45}\mu + \beta_{44},
\]

\[
\beta_{mn} = a_{mn} - a_{m3}/a_{33} \quad (m, n = 1, 3),
\]

\[
\lambda_j = -l_3(\mu_j)/l_2(\mu_j), \quad \lambda_3 = -l_3(\mu_3)/l_4(\mu_3);
\]

2774