FREE OSCILLATIONS OF A MULTILAYER STRATIFIED LIQUID SEPARATED BY ELASTIC MEMBRANES

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Solutions are obtained for the eigenmodes of a multilayer ideal stratified liquid separated by elastic membranes. The limiting cases of zero stress on the membrane and large filling depths are studied. The cases of three- and two-layer stratified liquids are examined in detail. The effect of a load on the first eigenmode is studied as a function of the membrane stress, the density of the liquid, and the filling depth.

The interaction of an elastic membrane with a homogeneous liquid has been studied in [1-3]. The effect of an elastic bottom on the eigenfrequencies of a stratified liquid has been studied [4] and a frequency equation has been obtained [5] for two layers of liquid separated by an elastic membrane. The quenching of waves by surface membranes has also been studied [6]. In this paper, we generalize the results of [5] to the case of an m-layer liquid.

Let us consider a mechanical system of m ideal immiscible liquids with densities \( \rho_i \) (i = 1, 2, ..., m) which fill a rigid circular cylinder of radius \( a \) to depths of \( h_i \), respectively. Flexible membranes are stretched over the free surface of the upper liquid and along the interfaces of the stratified liquid with running stresses \( T_i \). The membranes are rigidly attached to the edge and are assumed to be weightless. We shall examine the motion of the liquids and membranes in a coordinate system \( Oxyz \) which is positioned so that the \( Oyz \) plane coincides with the free surface of the upper liquid in its unperturbed position and the \( Ox \) axis is directed along the axis of the cylinder. We shall solve the problem in a linear approximation and assume that the motion of the liquids is irrotational.

The boundary value problem for the motion in this mechanical system has the form

1. \( \Delta \Phi_i = 0 \) (i = 1, 2, ..., m);
2. \( \frac{\partial \Phi_i}{\partial r}|_{r=a} = 0 \); \( \frac{\partial \Phi_i}{\partial x}|_{x=H_{i+1}} = 0 \);
3. \( \frac{\partial \Phi_i}{\partial x} = \frac{\partial W_i}{\partial r}, \quad \frac{\partial \Phi_{i+1}}{\partial x} = \frac{\partial \Phi_i}{\partial x} (x = -H_i); \)
4. \( \frac{\partial^2 W_i}{\partial x^2} + \frac{1}{r} \frac{\partial W_i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 W_i}{\partial \theta^2} = \left( \rho_{i+1} - \rho_i \right) T_i (x = -H_i); \)
5. \( W_i|_{r=a} = 0; \quad P_i = -\rho_i \left( \frac{\partial \Phi_i}{\partial t} + gx - Q_i(t) \right). \)

Here \( H_1 = 0 \) and \( H_i = \sum_{k=1}^{i-1} h_k \); \( P_i \) is the excess pressure in the \( i \)th liquid; \( \Phi_i \) is the velocity potential of the liquid; \( W_i \) is the bending of the \( i \)th membrane; \( g \) is the acceleration of gravity; and, \( Q_i(t) \) is an arbitrary function of time.

Limiting ourselves just to the first mode in the angular coordinate, we shall seek a solution of Eqs. (1)-(5) in the form

\[ \Phi_i = \omega \cos \omega t \cos \theta \phi_i(x, r); \quad W_i = \sin \omega t \cos \theta w_i(r). \]

The form of the potential $\phi(x, r)$ which satisfies the system of equations (1) and (2) can be represented as follows:

$$
\phi_i = \sum_{n=1}^{\infty} \left( A_{in} e^{k_n x} + B_{in} e^{-k_n x} \right) R_n(k_n r) \quad (i = 1, 2, ..., m-1);
$$

$$
\phi_m = \sum_{n=1}^{\infty} A_{mn} \cosh k_n (r + H_{m+1}) R_n(k_n r),
$$

(7)

where

$$
R_n(k_n r) = J_1(k_n r)/J_1(\mu_n); \quad \mu_n = k_n a; \quad J_1'(\mu_n) = 0.
$$

We represent the function $w_i(r)$ in the analogous form

$$
w_i(r) = A_i r + \sum_{n=1}^{\infty} C_{in} R_n(k_n r).
$$

(8)

Substituting Eqs. (6)-(8) in Eqs. (3)-(5) and using the orthogonality of the Bessel function $J_1(k_n r)$, we obtain the following system of equations for the unknown $A_{in}, B_{in}, C_{in}$, and $A_i$:

$$
(A_{i-1} - A_i) e^{-k_n H_i} - (B_{i-1} - B_i) e^{k_n H_i} = 0 \quad (i = 2, ..., m-1),
$$

(9)

$$
A_{m-1} e^{-k_n H_m} - B_{m-1} e^{k_n H_m} = A_{mn} \sinh k_m n;
$$

$$
A_{in} e^{-k_n H_i} - B_{in} e^{k_n H_i} = a_{in} \quad (i = 1, ..., m-1),
$$

(10)

$$
A_{mn} \sinh k_m n = \delta_{mn};
$$

$$
k_n^2 T_i C_{in} = \left( a_{in} A_{i-1} - \omega^2 \rho_{i-1} (A_{i-1}) \right) e^{-k_n H_i} -
$$

$$
- \left( \beta_{in} B_{i-1} + \omega^2 \rho_{i-1} (B_{i-1}) \right) e^{k_n H_i} \quad (i = 1, ..., m-1),
$$

(11)

$$
k_n^2 T_m C_{mn} = \left[ \omega^2 \rho_m \cosh \kappa_m - g\kappa_m (\rho_m - \rho_{m-1}) \right] \times
$$

$$
\times A_{mn} \sinh k_m n - \omega^2 \rho_{m-1} \left( A_{m-1} e^{-k_n H_m} + B_{m-1} e^{k_n H_m} \right);
$$

$$
A_i a + \sum_{n=1}^{\infty} C_{in} = 0 \quad (i = 1, ..., m),
$$

(12)

where

$$
\bar{a}_{in} = \left( 2 A_i a / \left( \mu_n^2 - 1 \right) + C_{in} \right) / k_n;
$$

$$
\bar{a}_{in} = \omega^2 \rho_i - g(\rho_i - \rho_{i-1}) k_n;
$$

$$
\bar{\beta}_{in} = - \omega^2 \rho_i - g(\rho_i - \rho_{i-1}) k_n \quad \kappa_{in} = k_n h_i.
$$

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