Some New Domains with Complete Kähler Metrics of Negative Curvature

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Abstract. We add to the known examples of complete Kähler manifolds with negative sectional curvature by showing that the following three classes of domains in Euclidean spaces also belong: perturbations of ellipsoidal domains in $\mathbb{C}^n$, intersections of complex-ellipsoidal domains in $\mathbb{C}^2$, and intersections of fractional linear transforms of the unit ball in $\mathbb{C}^2$. In the process, we prove the following theorem in differential geometry: in the intersection of two complex-ellipsoidal domains in $\mathbb{C}^2$, the sum of the Bergman metrics is a Kähler metric with negative curvature operator.

The purpose of this paper is to add to the existing list of complete Kähler manifolds of negative (sectional) curvature by exhibiting several domains in complex Euclidean spaces which admit such Kähler metrics. Before discussing our results, it is necessary to first review the current state of affairs along this line.

It is a mildly startling fact that, in spite of the importance of complete Kähler metrics of negative curvature in geometry and complex function theory, a fact so convincingly demonstrated since Poincaré's discovery of his metric on the unit disc, and in spite of the considerable interest in such metrics in the past decades, the only known example until around 1980 of a complete simply connected Kähler manifold of negative curvature in dimension $n$ for $n > 1$ was the unit ball in $\mathbb{C}^n$. This fact led some to the conjecture that if a complete simply connected Kähler manifold has negative curvature and if it has a compact quotient, then it must be biholomorphic to the unit ball. (We recall that the Bergman metric on the unit ball is a complete Kähler metric of holomorphic curvature equal to $-4$, and hence its sectional curvature lies between $-4$ and $-1$.) In particular,
this shows that the need of concrete examples of such manifolds is very real. Depending on how one interprets "negative" then, the following seem to be all the known examples up to now:

**Example 1.** Mostow and Siu constructed in [12] a two-dimensional compact Kähler manifold of negative curvature whose universal covering cannot be the unit ball for topological reasons. This then settles the above conjecture negatively. The complex curvature operator of this metric is in fact negative definite (see Section 2 for the precise definition). At the moment, it is still unknown if this universal covering is biholomorphic to any bounded domain in $\mathbb{C}^2$.

**Example 2.** All $C^4$ deformations of the unit ball admit complete Kähler metrics of uniformly negative curvature; moreover, the Bergman metrics of these domains are also complete Kähler metrics of uniformly negative curvature (strictly speaking, the validity of this fact would require a $C^k$-deformation for some $k$ larger than 4). These two facts were proved by Greene and Krantz in [7] and [8] respectively. Again, the complex curvature operators of these metrics are negative definite. Note that by [4], these deformations are generically holomorphically distinct from the ball and from each other.

**Example 3.** Let $p > 1$ and let $D_p$ be the Thullen domain in $\mathbb{C}^2$ defined by $\{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^{2p} < 1\}$. In [1] and [2], K. Azukawa and M. Suzuki computed the holomorphic curvature of the Bergman metric of $D_p$. It follows easily from their work (e.g., p. 5 of [2]) that if $p$ is, say, strictly between 1 and 2.5, then the curvature of the Bergman metric of $D_p$ is uniformly negative. Their calculations, however, did not give information on the negative definiteness of the curvature operator. Recall that $D_p$ is only weakly pseudoconvex along the circle on the boundary defined by $w = 0$ and $z = 1$. Note that for $1 < p < 1.5$, $D_p$ is $C^2$ but not $C^3$.

**Example 4.** In a work that parallels that of Azukawa and Suzuki in Example 3, J. S. Bland [3] computed the holomorphic curvature of the canonical Einstein–Kähler metric $E$ (of Cheng et al. [5]) on the Thullen domains $D_p$ in $\mathbb{C}^n$, where $|z|^2$ now has to be interpreted as the norm squared of the point $z = (z^1, \ldots, z^{n-1}) \in \mathbb{C}^{n-1}$. His conclusion is that for all $p > 1$, $E$ has uniformly negative sectional curvature. Moreover, the complex curvature operator of $E$ is negative definite.

**Example 5.** A domain $D$ of $\mathbb{C}^n$ is called an *ellipsoidal domain* if it is the interior of an ellipsoid in $\mathbb{R}^{2n}$, where the latter refers to the underlying real euclidean space of $\mathbb{C}^n$. In 1987, A. Nannicini [13] proved by a direct construction that all ellipsoidal domains admit complete Kähler metrics of uniformly negative curvature. It is not known whether the complex curvature operator of this metric is negative definite. We remark that by the work of [15], these ellipsoidal domains are holomorphically distinct from the unit ball except for the obvious cases.

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2"Uniformly negative" means bounded above by a negative constant. This terminology is standard in other parts of mathematics (e.g., uniformly elliptic, uniformly bounded) so we see no reason why the geometers should not take it up.