Cartan’s Magic Formula and Soap Film Structures

By Jenny Harrison

ABSTRACT. A soap film is actually a thin solid fluid bounded by two surfaces of opposite orientation. It is natural to model the film using one polyhedron for each side. Two problems are to get the polyhedra for both sides to be in the same place without canceling each other out and to model triple junctions without introducing extra boundary components. We use chainlet geometry to create dipole cells and mass cells which accomplish these goals and model faithfully all observable soap films and bubbles. We introduce a new norm on chains of these cells and prove lower semicontinuity of area. A geometric version of Cartan’s magic formula provides the necessary boundary coherence.

1. Introduction

Plateau observed that soap films have only two possible kinds of branching: (1) three sheets of surface meeting at 120° angles along a curve and (2) four such curves meeting at approximately 109° angles at a point. The question known as Plateau’s Problem naturally arose,

Given a loop of wire in 3-space, is there a surface with minimal area spanning it?

Osserman [12] wrote “the question of how to formulate this problem precisely has been almost as much of a challenge as devising methods of its solution.” Over the past century several mathematical models for area minimizing surfaces have been produced. The regularity and singularity structure of the resulting surfaces have been extensively studied, but none are sufficiently general to include all types of surfaces that arise as soap films.

The first solution to Plateau’s Problem was due to J. Douglas [4] who established the existence of area-minimizing disks for a given Jordan curve. He showed that amongst all mappings of the disk into $\mathbb{R}^3$ such that the disk boundary is mapped homeomorphically onto a given curve $\gamma$, there exists at least one mapping whose image has smallest possible area. Douglas was awarded one of the first Fields’ medals for this work.

Federer and Fleming [5] proved the existence of area-minimizing embedded orientable surfaces. They found an integral current with minimal mass spanning a given curve. Fleming later

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proved that this integral current was an embedded, orientable surface. Almgren used varifolds [2] to treat the nonorientable case but found no natural boundary operator. Soap film regularity of varifolds remains an open question. Brakke [3] produced interesting models of soap films using a covering space of the complement of the soap film boundary. There remain questions about existence and regularity of area minimizers in this category and it is not known whether all soap films are modeled.

Using methods and results from chainlet geometry [7, 8] summarized in Section 2, we define new models for soap films called dipolyhedra in Section 3, giving new definitions to the notions of curve, surface, and area. In Section 4 we prove lower semicontinuity of area and in Section 5 we show that dipolyhedral models apply to all soap films observed by Plateau, soap bubbles, the surfaces studied by Douglas, Federer, and Fleming, as well as films with singular branched curves, nonorientable films, films touching only part of the Jordan curve boundary and other examples that have eluded previous models.

FIGURE 1

2. Normed groups of polyhedra

The concept of continuity of domains plays an important role in our approach. In the first two sections we develop the concept of closeness for domains. This is conveniently done by starting with norms on polyhedra.

For $k \geq 1$, a $k$-dimensional cell in $\mathbb{R}^k$ is defined to be the finite intersection of open half spaces. A $k$-cell in $\mathbb{R}^n$ is a subset of a $k$-plane $\Pi$ in $\mathbb{R}^n$ that is a $k$-cell in $\Pi$. All $k$-cells are assumed to be oriented. A 0-cell is a single point $\{x\}$ in $\mathbb{R}^n$. No orientation need be assigned to the 0-cell. One may take formal sums of $k$-cells with integer coefficients $S = \sum a_i \sigma_i$ and form equivalence classes of polyhedral chains $P = [S]$. Two formal sums are equivalent if integrals

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