A note on the Asymptotic Generalised Variance for a Moving Average process

Summary

Relationships between the asymptotic generalised variance, of an \( r \)th order moving average process, and the ordinary variance, of an associated \( r \)th order autoregressive process, are established for \( r=1 \) and 2. This is extended to the case \( r=3 \), and a generalisation is suggested for all \( r>0 \), though this has not been checked beyond \( r=4 \).

Some key words: Moving average process, Autoregressive process, Invertibility, Autocovariance, Autocovariance matrix, Generalised variance, Autocorrelation.

1. Introduction

Consider the moving average process of order \( r \)

\[
Z_i = \sum_{j=0}^{r} \theta_j A_{i-j} \tag{1.1}
\]

with \( \theta_0 = 1 \); where the \( A_{i-j} \) are uncorrelated random variables, with zero mean and unit variance and all identically distributed. We will assume that (1.1) is invertible, which implies that the polynomial \( \sum_{j=0}^{r} \theta_j \zeta^j \), in the complex variable \( \zeta \), has all its zeros strictly outside the unit circle.

The autocovariance at lag \( \ell \), for (1.1), is defined by

\[
\gamma_{\ell} = \text{Cov}[Z_i, Z_{i-\ell}]
\]

which is independent of \( i \). In fact, for \( \ell \geq 0 \),

\[
\gamma_{\ell} = \sum_{j=0}^{r-\ell} \theta_j \theta_{j+\ell}.
\]

For any positive \( k \), the \( k \times k \) autocovariance matrix of (1.1) is defined as

\[
P_k = (p_{st})
\]
where

\[ P_{st} = Y|s-t|. \]

We will denote the determinant of \( P_k \) by \( D_k \), which is the \( k \)th generalised variance, and define \( D_0 \) to be one.

Associated with (1.1) is an \( r \)th order autoregressive process

\[ \sum_{j=0}^{r} \theta_j z_{i-j} = A_i \]  

(1.2)

whose autocorrelation at lag \( \lambda \) is defined by

\[ \rho_\lambda = \frac{\text{Cov}[Z_i, Z_{i-\lambda}]}{\text{Var}[Z_i]}. \]

Then \( \rho_0 = 1 \) and

\[
\begin{bmatrix}
\rho_1 \\
\vdots \\
\rho_r
\end{bmatrix} = 
\begin{bmatrix}
1 & \rho_1 & \cdots & \rho_{r-1} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots \\
\rho_r & \cdots & \rho_1 & 1
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\vdots \\
\theta_r
\end{bmatrix}
\]

and a sequence \( \{z_i\} \), following (1.2), has its variance given by

\[ \nu_r = \left( \sum_{j=0}^{r} \rho_j \theta_j \right)^{-1}. \]

So, for instance,

\[ \nu_1 = (1-\theta_1^2)^{-1} \]  

(1.3)

\[ \nu_2 = (1 + \theta_2)^2 - \theta_1^2 \left( \frac{1}{1 - \theta_2} \right) \]  

(1.4)

Finally we introduce the shift operator \( B \), such that for any sequence of determinants, \( \{F_k\} \) say,

\[ B^j F_k = F_{k-j} \]

for all integers \( j \).