SOME RESULTS ON ALMOST-PERIODIC DIFFERENTIAL EQUATIONS

(Conferenza tenuta il 27 maggio 1971) *

SUNTO. — Si espongono alcuni risultati recenti sull'argomento indicato nel titolo.

INTRODUCTION.

Some time ago, Bohr-Neugebauer and Bochner proved the following result (see [2]).

«If \( f(t) = (f_1(t), \ldots, f_n(t)) \) is an almost-periodic vector-valued function; if \( A = (a_{ij})_{i,j=1}^n \) is a constant \( n \times n \) matrix; if \( u(t) = (u_1(t), \ldots, u_n(t)) \) is a bounded on \(-\infty < t < \infty\) vector-valued function; and if the system

\[
\frac{du(t)}{dt} = Au(t) + f(t), \quad -\infty < t < \infty
\]

is satisfied, then \( u(t) \) is also almost-periodic ».

Recently various results of the same type were obtained, i.e. « boundedness implies almost-periodicity » (look for example in Amerio-Prouse monograph [1]).

Some other wait still for publication; one is due to the author [4], and will be explained below.

Before that, however, we shall remember that we had the following result (see [3], where unbounded operators are also considered).

«If \( H \) is a hilbert space; \( A \in L(H;H) \) is a linear bounded self-adjoint operator in \( H \); \( f(t), u(t), -\infty < t < \infty \rightarrow H \) are res-

* Pervenuta in tipografia il 9 luglio 1971.
pectively almost-periodic and bounded; the equality \( u'(t) = Au(t) + f(t), -\infty < t < \infty \), holds; then \( u(t) \) is almost-periodic too.

Here almost-periodicity is taken in Bohr's sense: i.e., the set of numbers \( \{ \tau \} \), such that \( \sup_{-\infty < t < \infty} \| f(t + \tau) - f(t) \| < \varepsilon \), for arbitrary given \( \varepsilon > 0 \), is relatively dense in \( -\infty < t < \infty \).

The theorem is well-known and has a trivial proof when \( A \) is strictly positive or strictly negative.

For \( A = 0 \) we get the known theorem on integrals of almost-periodic functions in hilbert spaces (see [1]).

Recently we tried to extend this result to the case where \( A \) is not necessarily self-adjoint.

Remark however that \( A \) cannot be too far from a self-adjoint operator; for example, \( A = \lambda B \), where \( B \) is self-adjoint, is sometimes an easily verified counter-example (take \( f(t) = \theta \); all solutions of the equation \( u'(t) = iBu(t) \) are bounded, of the form \( u(t) = e^{iBt} u(0) \); they are not necessarily almost-periodic).

1. - Let us define our «admissible» class of operators \( A \) in [4]; Take \( A = A_0 + B \) where:

\( A_0 \) is linear bounded self-adjoint operator in \( H \) and \( \lambda = 0 \) is not an eigen-value for \( A_0 \); in the same time, \( B \) is a « nice » bounded perturbation.

We assume precisely that \( BA_0 = A_0 B \), and without essential loose of generality we shall take \( 0 \leq A_0 \leq I \); furthermore, if \( (E_\lambda)_{\lambda=-\infty}^{\infty} \) is the spectral family for \( A_0 \) and if \( F_n = E_\frac{1}{n} - E_\frac{1}{n+1} \) we assume that

\[
\sup_{-\infty < \lambda < \infty} \| e^{iB\lambda t} \| = A_n < + \infty, \quad n = 1, 2, \ldots \quad \text{and}
\]

\[
\sum_{n=1}^{\infty} \| B F_n \|_{L^2(H)} < + \infty.
\]

Then we have

**Theorem 1.** - If \( f(t) \) is \( H \)-valued almost-periodic function; if \( u(t) \) is \( H \)-valued bounded function; if

\[
u'(t) = (A_0 + B) u(t) + f(t), \quad -\infty < t < \infty
\]

then \( u(t) \) is \( H \)-almost-periodic.