Binary probit, tobit and hazard rate
A didactical note in microeconometrics
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This short note draws attention to an obvious although so far only partly noted relation between an inequality involving the hazard rate of the normal distribution and maximum likelihood estimation in the binary probit and in the Tobit model. Global concavity of the likelihood function for both the binary probit and the Tobit model can be proved by means of an inequality concerning the hazard rate of the standard normal distribution. As not yet noted in the literature this inequality may also be used to show that the hazard rate is monotonically increasing.

Keywords: Global concavity of likelihood function; truncated distribution.

In this short note we draw attention to an obvious although so far only partly noted relation between an inequality involving the hazard rate of the normal distribution and maximum likelihood estimation in the binary probit and in the Tobit model. Since the hazard rate usually is mentioned only in connection with duration models this note may be seen as an effort to bring closer together the different fields of microeconometrics. We first state the inequality and then show its importance for the binary probit model and the Tobit model. Some bibliographical remarks conclude the presentation.

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Let $V$ be a random variable which is truncated from below at $c$, i.e. $P(V > c) = 1$. Now assume that $V$, if not truncated, follows the standard normal distribution. Then the expected value of $V$ is given by

$$E(V | V > c) = \frac{\phi(c)}{1 - \Phi(c)}$$

(Maddala 1983 p. 365). Now from the fact that $V$ is truncated at $c$ we immediately obtain

$$\frac{\phi(c)}{1 - \Phi(c)} > c, \quad c > 0$$

Inequality (2) can be interpreted as an inequality $q(c) > c$ concerning the hazard rate $q$. Note that this inequality or rather its generalization mentioned in footnote 1 shows that for the normal distribution the hazard rate is monotonically increasing.³

We now turn to the binary probit model which is defined as follows: A dichotomous observable random variable $Y$ takes value 1 if a latent variable $Y^*$ takes values greater than zero where $Y^* \sim N(\mu, \sigma^2)$ and $\mu = x'\beta$ where $x$ and $\beta$ are $K$-dimensional vectors. Then we have

$$P(Y = 1 | x) = \Phi(x'\beta)$$

where $\Phi$ denotes the distribution function of the standard normal distribution.

Let $y_t$ be the outcome of the $t$-th observation of $Y$ and denote by $x_t$ the corresponding vector of explanatory variables where $t = 1, \ldots, n$. The matrix of second order partial derivatives of the loglikelihood function is given by

$$\frac{\delta^2 \log L}{\delta \beta \delta \beta'} = -\sum_{t} \frac{\phi_t}{\Phi_t(1 - \Phi_t)^2} g(y_t) x_t x_t'$$

where

$$g(y_t) = (y_t - 2y_t\Phi_t + \Phi_t^2)\phi_t + (y_t - \Phi_t)\Phi_t(1 - \Phi_t)x_t$$

¹If $V \sim N(\mu, \sigma^2)$ then the right-hand side of (1) is changed to $\mu + \sigma\phi(c)/(1 - \Phi(c))$. See, for example, Amemiya (1985 p. 367).

²Obviously the inequality is nontrivial only for $c > 0$.

³It is easily verified that the derivative of $q(c)$ is proportional to $\phi(c) - c(1 - \Phi(c))$ and therefore positive.

For a discussion of the parametric family of distributions with monotone hazard rate see Barlow and Proschan(1975, section 3.5) who also provide an alternative proof for the above result.