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POSITIVE ENERGY IN FIELD THEORIES
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SUNTO. — In questo lavoro si ottengono condizioni sufficienti a poter garantire la positività dell'energia per una classe di teorie di campo. Tale classe è caratterizzata dalla deducibilità delle equazioni di campo da un principio variazionale simile a quello da cui si possono ottenere le equazioni di Einstein. Si tratta, cioè, di teorie che si valgono di uno spazio Lorentziano n-dimensionale $V_n$, il cui tensore metrico $\gamma_{\alpha\beta}$ individua una curvatura scalare $B$ e per le quali il principio variazionale richiede che la metrica $\gamma_{\alpha\beta}$ sia tale da rendere stazionario l'integrale di $B$ esteso a $V_n$. Rientrano nel predetto schema la teoria pentadimensionale di Kaluza-Klein e quella di Jordan-Thiry, come pure le generalizzazioni di tali teorie che interessano la fisica delle particelle. Viene messo in evidenza che le soluzioni statiche delle equazioni (1.9) ottenute, per $n = 5$, da Sorkin e da Gross e Perry non soddisfano le ipotesi assunte da Einstein e da Lichnerowicz per la loro soluzione stazionaria di dette equazioni.

1. - INTRODUCTION.

The field theories with which we shall be concerned will be those that are described by an $n$-dimensional Lorentzian metric tensor (one of signature $-, +, +, \ldots +$), $\gamma_{\alpha\beta} (\alpha, \beta = 1, 2, \ldots, n)$, for which the field equations are derived from a variational principle involving the integral over the $n$-dimensional space of the scalar curvature formed from this tensor. Such field theories include the Einstein theory of gravitation, the five-dimensional classical unified field theories of Kaluza-Klein, and that of Jordan-Thiry [1], as well as the higher dimensional Kaluza-Klein [2] theories. In the latter theories the metric $\gamma_{\alpha\beta}$ is introduced by considering the right invariant metric on the principal fibre bundle over four dimensional spacetime for which the vertical and horizontal tangent spaces of the bundle are orthogonal to each other with respect to $\gamma_{\alpha\beta}$.

It may be shown that coordinates $x^a (a = 1, 2, \ldots, n)$ may be introduced in the fibre bundle in which

\begin{equation}
\gamma_{\alpha\beta} dx^\alpha dx^\beta = g_{ij}(x^k) dx^i dx^j + g_{pq}(x^\omega) \tilde{\omega}^p \tilde{\omega}^q
\end{equation}
where

\begin{align}
    i, j, k &= 1, 2, 3, 4 \\
    p, q &= 5, 6, \ldots, n
\end{align}

(1.2)

\[ \tilde{\omega}^p = \tilde{\theta}^p + A_i^p \, dx^i, \]

\( \tilde{\theta}^p \) are right invariant one forms on the \( n - 4 \) dimensional group which is isomorphic to the vertical spaces (the fibres) of the principal bundle and \( A_i^p \) are the connection one forms of the bundle. We shall assume that \( n = 2\nu + 1 \). The case for which \( n = 2\nu \) may be treated by considering a special hypersurface in the former case.

The Jordan-Thiry classical unified field theory is an example of the class of theories with which we shall be concerned. In it \( n = 5 \) and the \( g_{ij} \) are interpreted as the metric tensor of a four-dimensional space time containing an electromagnetic field determined by \( \gamma_{5i}/\gamma_{55} \) and \( \gamma_{55} \) is a scalar field related to a varying « gravitational constant ».

The original Kaluza-Klein theory and that of Jordan-Thiry were not formulated as gauge theories, that is, they were not considered as fibre bundles over a Lorentzian four-dimensional space. Instead each was considered as a five-dimensional Lorentzian space \( V_5 \) admitting a spacelike Killing vector. It was assumed that coordinates could be introduced so that the Killing vector had the form

(1.3) \[ \xi^a = \delta_5^a \]

globally. Then one may write

\[ ds^2 = \gamma_{\alpha\beta} \, dx^\alpha \, dx^\beta = \gamma_{ij} \, dx^i \, dx^j + 2\gamma_{5i} \, dx^i \, dx^5 + \gamma_{55} \,(dx^5)^2 \]

where the \( \gamma_{\alpha\beta} \) are independent of \( x^5 \). This equation may in turn be written as

(1.4) \[ ds^2 = g_{ij} \, dx^i \, dx^j + W^s (dx^5 + A_i \, dx^i)^2 \]

\[ g_{ij} = \gamma_{ij} - (\gamma_{55})^{-1} (\gamma_{5i} \gamma_{5j}), \]

(1.5) \[ W^s A_i = \gamma_{5i} \]

\[ W^s = \gamma_{55}. \]

It is assumed that the above decomposition is global, that is,

(1.6) \[ V_5 = V_4 \times \mathbb{R}. \]