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A STREAMLINE DIFFUSION FINITE ELEMENT METHOD ON A SHISHKIN MESH FOR A CONVECTION-DIFFUSION PROBLEM

Conferenza tenuta il 13 luglio 1994

ABSTRACT. We consider a model time-dependent convection-diffusion problem, whose solution may exhibit interior and boundary layers. The standard streamline diffusion scheme, with piecewise linear elements on a uniform mesh, will converge only at points that are not close to any layer. We replace the uniform mesh by a special piecewise uniform mesh that is chosen a priori and resolves part of any outflow boundary layer. The resulting method is convergent, independently of the diffusion parameter, with a pointwise accuracy of almost order 5/4 away from layers and almost order 3/4 inside the boundary layer.

1. Introduction

We consider the model time-dependent convection-diffusion problem

\[
\begin{align*}
-\varepsilon u_{xx} + au_x + u + u_t &= f(x,t) \quad \forall (x,t) \in \Omega, \\
\varepsilon u(0,t) &= u(1,t) = 0 \quad \text{for } 0 < t \leq T, \quad (1.1) \\
u(x,0) &= u_0(x) \quad \text{for } 0 \leq x \leq 1,
\end{align*}
\]

where \( \Omega = (0,1) \times (0,T] \), \( \varepsilon \) is a small positive parameter, \( a > 0 \) is a constant, and \( u_0 \in L^2[0,1] \), \( f \in L^2(\Omega) \). Here for simplicity we have taken the coefficients of the differential equation to be constant; the result below still hold for smooth variable coefficients provided that they remain positive on \( \Omega \).

The solution \( u \) of (1.1) has in general a boundary layer along the side \( x = 1 \) of \( \Omega \). By a "layer" we mean a narrow region where some first-order derivative of \( u \) is large (typically \( O(1/\varepsilon) \)). If the initial data
$u_0(x)$ has a jump discontinuity, this will cause an *interior layer* in $u$. In this case the layer lies along a characteristic trace of the first-order hyperbolic problem

\[
\begin{cases}
av_x + v + v_t = f(x,t), & \forall (x,t) \in \Omega, \\
v(0,t) = 0, & \text{for } 0 < t \leq T, \\
v(x,0) = u_0(x), & \text{for } 0 \leq x \leq 1.
\end{cases}
\]

We call (1.2) the *reduced problem*. At points that are not close to any layer, $u$ is essentially indistinguishable from $v$, the solution of the reduced problem.

It is well-known that (1.1) is difficult to solve numerically; see, e.g., Johnson [4] or Roos et al. [8]. For example, if we use a space-centred explicit Euler discretization on a uniform mesh with mesh spacings $\Delta x$ and $\Delta t$, then $\Delta t = O(\varepsilon)$ is a necessary condition for stability [2].

Various numerical techniques have been proposed for the solution of (1.1), but here we restrict our attention to the well-known streamline diffusion finite element method. This method was devised for convection-diffusion problems. It is usually accurate away from layers, but is less satisfactory in regions where $u$ changes rapidly. The purpose of this paper is to describe an implementation of the streamline diffusion method on a special mesh for which one can prove robust error estimates both away from layers and inside a typical boundary layer. No such result for this method has previously appeared in the literature.

Throughout the paper, $C$ (sometimes subscripted) will denote a generic positive constant, not necessarily the same at each occurrence, that is independent of $\varepsilon$ and of any mesh used.

2. The Streamline Diffusion Method

We begin by describing the streamline diffusion (SD) finite element method on a uniform triangular mesh.

Let $N$ and $M$ be two positive integers, satisfying $\max\{N/M, M/N\} \leq C$. We shall always assume that $\varepsilon \leq N^{-1}$, which is reasonable in